

SPECTRAL INVARIANTS WITH BULK, QUASIMORPHISMS AND LAGRANGIAN FLOER THEORY

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ABSTRACT. In this paper we first develop various enhancements of the theory of spectral invariants of Hamiltonian Floer homology and of Entov-Polterovich theory of spectral symplectic quasi-states and quasimorphisms by incorporating *bulk deformations*, i.e., deformations by ambient cycles of symplectic manifolds, of the Floer homology and quantum cohomology. Essentially the same kind of construction is independently carried out by Usher [Us4] in a slightly less general context. Then we explore various applications of these enhancements to the symplectic topology, especially new construction of symplectic quasi-states, quasimorphisms and new Lagrangian intersection results on toric manifolds

The most novel part of this paper is to use open-closed Gromov-Witten-Floer theory (operator \mathfrak{q} in [FOOO1] and its variant involving closed orbits of periodic Hamiltonian system) to connect spectral invariants (with bulk deformation), symplectic quasi-states, quasimorphism to the Lagrangian Floer theory (with bulk deformation).

We use this open-closed Gromov-Witten-Floer theory to produce new examples. Especially using the calculation of Lagrangian Floer homology with bulk deformation in [FOOO2, FOOO3], we produce examples of compact toric manifolds (M, ω) which admits uncountably many independent quasimorphisms $\widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$.

We also obtain a new intersection result of Lagrangian submanifold on $S^2 \times S^2$ discovered in [FOOO5].

Many of these applications were announced in [FOOO2, FOOO3, FOOO5].

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1. INTRODUCTION

Let (M, ω) be a compact symplectic manifold.

We consider one-periodic nondegenerate Hamiltonians $H : S^1 \times M \rightarrow \mathbb{R}$, *not necessarily* normalized, and one-periodic family $J = \{J_t\}_{t \in S^1}$ of almost complex structures compatible with ω . To each given such pair (H, J) , we can associate the Floer homology $HF(H, J)$ by considering the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial \tau} + J_t \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0, \quad (1.1)$$

where $H_t(x) = H(t, x)$ and X_{H_t} is the Hamiltonian vector field associated to $H_t \in C^\infty(M)$. The associated chain complex $(CF(M, H), \partial_{(H, J)})$ is generated by the pairs $[\gamma, w]$ where γ is a loop satisfying $\dot{\gamma}(t) = X_{H_t}(\gamma)$, $w : D^2 \rightarrow M$ is a disc with $w|_{\partial D^2} = \gamma$ and $[\gamma, w]$ is the homotopy class relative to the boundary γ . This chain complex carries a natural downward filtration provided by the action functional

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int_0^1 H(t, \gamma(t)) dt, \quad (1.2)$$

since (1.1) is the negative L^2 -gradient flow of \mathcal{A}_H with respect to the L^2 -metric on $\mathcal{L}(M)$ defined by

$$\int_0^1 \langle \xi_1(t), \xi_2(t) \rangle_J dt$$

where $\langle \xi_1(t), \xi_2(t) \rangle_J = \omega(\xi_1(t), J_t \xi_2(t))$.

The homology group of $(CF(M, H), \partial_{(H, J)})$, the Floer homology associated to the one-periodic Hamiltonian H , is known to be isomorphic to the ordinary homology of M with respect to an appropriate Novikov ring coefficients ([Fl]).

The spectral invariants constructed by the second named author in [Oh4] for the general non-exact case are defined as follows. (See [Vi1, Oh1, Sc2] for the earlier related works for the exact case.) First take the mini-max value

$$\mathbf{v}_q(\alpha) = \max\{\mathcal{A}_H([\gamma_i, w_i]) \mid \alpha = \sum a_i [\gamma_i, w_i], a_i \in \mathbb{C} \setminus \{0\}\}, \quad (1.3)$$

$$\rho((H, J); a) = \inf\{\mathbf{v}_q(\alpha) \mid \partial_{(H, J)}(\alpha) = 0, [\alpha] = a\}, \quad (1.4)$$

(where $a \in HF(H, J) \cong H(M)$) and then prove that $\rho((H, J); a)$ does not depend on the choice of J . The spectral invariant $\rho(H; a)$ is nothing but this common value of $\rho((H, J); a)$ for the nondegenerate Hamiltonian H . Via the C^0 -continuity of the function $H \mapsto \rho(H; a)$, the function extends continuously to arbitrary continuous function H .

Let denote by $\text{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of M and by $\widetilde{\text{Ham}}(M, \omega)$ its universal cover. We denote by $\phi_H : t \mapsto \phi_H^t$ the Hamiltonian path (based at the identity) generated by the (time-dependent) Hamiltonian H and its time one map by $\psi_H = \phi_H^1 \in \text{Ham}(M, \omega)$. Each Hamiltonian H generates the Hamiltonian path ϕ_H which in turn determines an element $\tilde{\psi}_H = [\phi_H] \in \widetilde{\text{Ham}}(M, \omega)$. Conversely, each smooth Hamiltonian path $[0, 1] \rightarrow \text{Ham}(M, \omega)$ based at the identity is generated by a unique *normalized* Hamiltonian H , i.e., H satisfying

$$\int_M H_t \omega^n = 0. \quad (1.5)$$

It is proved in [Oh4], [Oh6] that $\rho(H; a)$ for normalized Hamiltonians H depends only on the homotopy class $\tilde{\psi} = \tilde{\psi}_H$ of the path ϕ_H and a , which we denote by $\rho(\tilde{\psi}; a)$. This homotopy invariance is proved for the rational (M, ω) in [Oh4] and for the irrational case in [Oh6], [Us1] respectively.

In a series of papers [EP1, EP2, EP3], Entov and Polterovich discovered remarkable applications of these spectral invariants to the theory of symplectic intersections and to the study of $\text{Ham}(M, \omega)$ by combining ideas from dynamical systems, function theory and quantum cohomology. We briefly summarize their construction now.

Entov and Polterovich [EP2] use the action functional

$$\tilde{\mathcal{A}}_H([\gamma, w]) = - \int w^* \omega + \int_0^1 H(t, \gamma(t)) dt$$

instead of (1.2) to define the spectral invariant $\rho^{EP}(H; 1)$. See Remark 4.17. They considered the function $\zeta_1^{EP} : C^0(M) \rightarrow \mathbb{R}$ first defined by

$$\zeta_1^{EP}(H) := \lim_{n \rightarrow \infty} \frac{\rho^{EP}(nH; 1)}{n} \quad (1.6)$$

for C^∞ function H and then extended to $C^0(M)$ by continuity. They proved in [EP1, EP2] that ζ_1^{EP} largely satisfies most of the properties of *quasi-states* introduced by Aarnes [Aa] and introduced the notion of *partial symplectic quasi-state*. In a more recent paper [EP3], they generalized the construction by incorporating other idempotent elements e of $QH^*(M; \Lambda)$, i.e., those satisfying $e^2 = e$, and also formulated the notions of *heavy* and *super-heavy* subsets of symplectic manifolds. Consider the (small) quantum cohomology ring $QH^*(M; \Lambda)$, where the coefficient ring is Λ that is the field of fractions of the universal Novikov ring

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\}.$$

(In fact, $\Lambda = \Lambda_0[T^{-1}]$.) When Entov-Polterovich's construction of partial symplectic quasi-states is carried out for an idempotent $e \in QH^*(M; \Lambda)$, we denote the corresponding partial symplectic quasi-states by $\zeta_e^{EP} = \zeta_e^{EP}(H) : C^0(M) \rightarrow \mathbb{R}$.

Entov and Polterovich also considered the function $\mu^{EP} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$ defined by

$$\mu^{EP}(\tilde{\psi}) := -\text{vol}_\omega(M) \lim_{n \rightarrow \infty} \frac{\rho^{EP}(\tilde{\psi}^n; 1)}{n}. \quad (1.7)$$

Similarly as μ^{EP} in (1.7), we associate to e the map $\mu_e^{EP} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$ by

$$\mu_e^{EP}(\tilde{\psi}) = -\text{vol}_\omega(M) \lim_{n \rightarrow \infty} \frac{\rho^{EP}(\tilde{\psi}^n; e)}{n}. \quad (1.8)$$

Whenever e is a unit of the direct factor of $QH^*(M; \Lambda)$, which is a field, μ_e^{EP} becomes a homogeneous quasimorphism. Namely it satisfies

$$\mu_e^{EP}(\tilde{\psi}_1) + \mu_e^{EP}(\tilde{\psi}_2) - C < \mu_e^{EP}(\tilde{\psi}_1 \tilde{\psi}_2) < \mu_e^{EP}(\tilde{\psi}_1) + \mu_e^{EP}(\tilde{\psi}_2) + C, \quad (1.9)$$

for some constant C independent of $\tilde{\psi}_1, \tilde{\psi}_2$ and

$$\mu_e^{EP}(\tilde{\psi}^n) = n \mu_e^{EP}(\tilde{\psi}), \quad \text{for } n \in \mathbb{Z}. \quad (1.10)$$

These facts were proved by Entov-Polterovich [EP1] in case $QH^*(M; \Lambda)$ is semi-simple and M is monotone. The monotonicity assumption was somewhat improved

by Ostrover [Os2]. It was observed by McDuff that instead of the semi-simplicity assumption one has only to assume that e is a unit of a factor of $QH^*(M; \Lambda)$ that is a field. In fact, Entov and Polterovich prove several other symplectic properties of μ_e^{EP} , and call them *Calabi quasimorphisms*.

The relationship between μ_e^{EP} and ζ_e^{EP} is as follows. Note that by definition we have $\rho^{EP}(\tilde{\psi}; e) := \rho^{EP}(\underline{H}; e)$ for a (and so any) Hamiltonian H such that $\tilde{\psi} = [\phi_H]$ where \underline{H} is the normalization of H which is given by

$$\underline{H}_t = H_t - \frac{1}{\text{vol}_\omega(M)} \int_M H_t \omega^n.$$

It follows from the action functional $\tilde{\mathcal{A}}_H$ and the mini-max values $\rho^{EP}(H; a)$ that

$$\begin{aligned} \rho^{EP}(\tilde{\psi}; e) &= \rho^{EP}(\underline{H}; e) = \rho^{EP}(H; e) + \frac{1}{\text{vol}_\omega(M)} \int_0^1 dt \int_M H_t \omega^n \\ &= \rho^{EP}(H; e) + \frac{1}{\text{vol}_\omega(M)} \text{Cal}(H) \end{aligned} \quad (1.11)$$

where

$$\text{Cal}(H) = \int_0^1 dt \int_M H_t \omega^n \quad (1.12)$$

is the Calabi invariant of H . (See Definition 13.2.) We also note that if H is autonomous Hamiltonian, we have $\phi_H^n = \phi_{nH}$. Therefore applying (1.11) to an *autonomous* Hamiltonian nH and its associated homotopy class $\tilde{\psi}^n = [\phi_{(nH)}]$ and dividing by n , one obtains

$$\frac{1}{n} \rho^{EP}(\tilde{\psi}^n; e) = \frac{1}{n} \rho^{EP}(nH; e) + \frac{1}{\text{vol}_\omega(M)} \text{Cal}(H).$$

By multiplying $\text{vol}_\omega(M)$ to this equation and taking the limit, we obtain the following identity

$$\mu_e^{EP}(\tilde{\psi}) = -\text{vol}_\omega(M) \zeta_e^{EP}(H) + \text{Cal}(H) = -\text{vol}_\omega(M) \zeta_e^{EP}(\underline{H}) \quad (1.13)$$

for any autonomous Hamiltonian H and its associated homotopy class $\tilde{\psi}$.

In Chapters 2 and 3 of this paper we modify the construction of spectral invariants, partial quasi-states, and quasimorphisms by involving the elements from the *big quantum cohomology ring* in Gromov-Witten theory. For this purpose, we deform the Floer homology $HF(H, J)$ by inserting an ambient (co)cycle \mathfrak{b} of even degree in the construction of boundary map, in exactly the same way as we did for the case of Lagrangian Floer theory in [FOOO1]¹.

Remark 1.1.

A similar construction has been also carried out by Usher [Us4] independently in a slightly less general context.

We denote the corresponding deformed Floer homology by $HF^{\mathfrak{b}}(H, J)$. Actually $HF^{\mathfrak{b}}(H, J)$ as a Λ module is also isomorphic to the ordinary homology group $H(M; \Lambda)$ of M . However it carries a filtration which contains certain new information. These constructions of spectral invariants and the associated spectral partial

¹Actually by considering the two Lagrangian submanifolds, the diagonal and the graph of time one map, the case of Hamiltonian diffeomorphism can be reduced to the case of Lagrangian submanifolds.

quasi-states and quasimorphisms with bulk can be generalized in a straightforward way except the following point:

We note that for the construction of partial quasi-states and quasimorphism the following triangle inequality of spectral invariant plays an important role.

$$\rho(\tilde{\psi}_1 \circ \tilde{\psi}_2, a \cup_Q b) \leq \rho(\tilde{\psi}_1, a) + \rho(\tilde{\psi}_2, b), \quad (1.14)$$

here \cup_Q is the product of the small quantum cohomology ring $QH^*(M; \Lambda)$ and ρ is the spectral invariant as defined in [Oh4] (without bulk deformation). Let us consider the spectral invariant with bulk, which we denote by $\rho^{\mathfrak{b}}(\tilde{\psi}, a)$. Then (1.14) becomes

$$\rho^{\mathfrak{b}}(\tilde{\psi}_1 \circ \tilde{\psi}_2, a \cup^{\mathfrak{b}} b) \leq \rho^{\mathfrak{b}}(\tilde{\psi}_1, a) + \rho^{\mathfrak{b}}(\tilde{\psi}_2, b), \quad (1.15)$$

where $\cup^{\mathfrak{b}}$ is the deformed cup product by \mathfrak{b} . (See Definition 5.1 for its definition.) Thus in place of the small quantum cohomology ring $QH^*(M; \Lambda)$ the \mathfrak{b} -deformed quantum cohomology ring (which we denote by $QH_{\mathfrak{b}}^*(M; \Lambda)$) plays an important role here.

Whenever $e \in QH_{\mathfrak{b}}^*(M; \Lambda)$ is an idempotent, we define

$$\zeta_e^{\mathfrak{b}}(H) = - \lim_{n \rightarrow \infty} \frac{\rho^{\mathfrak{b}}(nH; e)}{n} \quad (1.16)$$

for the autonomous function $H = H(x) \in C^\infty(M)$ which in turn defines a partial symplectic quasi-state on $C^0(M)$. See Definition 13.6 and Theorem 14.1. Similarly we can define $\mu_e^{\mathfrak{b}} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$. We will call any such partial quasi-state or quasimorphism obtained from spectral invariants as a whole *spectral partial quasi-state* or *spectral quasimorphism* respectively.

Remark 1.2. Note that we use the action functional \mathcal{A}_H , not $\tilde{\mathcal{A}}_H$, hence

$$\zeta_e(H) = -\zeta_e^{EP}(-H), \quad \mu_e(\tilde{\psi}_H) = -\mu_e^{EP}(\tilde{\psi}_{-H})$$

for the case $\mathfrak{b} = 0$ in our convention.

Then for any homotopy class $\tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$ generated by an autonomous Hamiltonian H , we have the equality

$$\mu_e^{\mathfrak{b}}(\tilde{\psi}) = \text{Cal}(H) - \text{vol}_\omega(M) \zeta_e^{\mathfrak{b}}(H). \quad (1.17)$$

Theorem 1.3. *Let $\Lambda e \cong \Lambda$ be a direct factor of $QH_{\mathfrak{b}}^*(M; \Lambda)$ and e its unit. Then*

$$\mu_e^{\mathfrak{b}} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$$

is a homogeneous Calabi quasimorphism.

Theorem 1.3 is proved in Section 16. In particular, combined with the study of big quantum cohomology of toric manifolds [FOOO6], this implies the following: (The proof is completed in Subsection 21.3.)

Corollary 1.4. *For any compact toric manifold (M, ω) , there exists a nontrivial homogeneous Calabi quasimorphism*

$$\mu_e^{\mathfrak{b}} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}.$$

We say that a quasimorphism is *nontrivial* if it is not bounded. Corollary 1.4 is also proved independently by Usher [Us4].

It is in general very hard to calculate spectral invariants and partial quasi-states or quasimorphisms obtained therefrom. In Chapter 4 of this paper we provide a means of estimating them in certain cases. We recall the following definition

Definition 1.5 (Entov-Polterovich [EP3]). Let $\zeta : C^0(M) \rightarrow \mathbb{R}$ be any partial quasi-state. A closed subset $Y \subset X$ is called ζ -heavy if

$$\zeta(H) \leq \sup\{H(p) \mid p \in Y\} \quad (1.18)$$

for any $H \in C^0(X)$. $Y \subset X$ is called ζ -superheavy if

$$\zeta(H) \geq \inf\{H(p) \mid p \in Y\} \quad (1.19)$$

for any $H \in C^0(X)$.

Due to the different sign conventions in the definitions of the action functional and $\zeta(H)$ used in [EP3] and in this paper, this definition looks opposite to that of [EP3]. However, by Remark 1.2 this definition is indeed equivalent to that in [EP3]. Entov-Polterovich proved in [EP3] Theorem 1.4 (i) that superheavyness implies heavyness for ζ_e . (See Remark 18.2.) The same can be proved for ζ_e^b by the same way.

We can also define a similar notion including time dependent Hamiltonian. See Definition 18.5.

Next we relate the theory of spectral invariants to the Lagrangian Floer theory. Let L be a relatively spin Lagrangian submanifold of M . In [FOOO1] we associated to L a set $\mathcal{M}_{\text{weak,def}}(L; \Lambda_+)$, which we call the *Maurer-Cartan moduli space*.

Remark 1.6. The Maurer-Cartan moduli space that appears in [FOOO1] uses the Novikov ring Λ_+ . A technical enhancement to its Λ_0 -version was performed in [FOOO3, Fu3] using the idea of Cho [Cho]. We include it in this paper. In this introduction, however, we state only the Λ_+ -version for the simplicity of exposition.

The Maurer-Cartan moduli space comes with a map

$$\pi_{\text{bulk}} : \mathcal{M}_{\text{weak,def}}(L; \Lambda_+) \rightarrow \bigoplus_k H^{2k}(M; \Lambda_+).$$

For each $\mathbf{b} \in \mathcal{M}_{\text{weak,def}}(L; \Lambda_+)$, the Floer cohomology $HF^*((L, \mathbf{b}), (L, \mathbf{b}); \Lambda_0)$ deformed by \mathbf{b} is defined in [FOOO1] Definition 3.8.61. Moreover the open-closed map

$$i_{\text{qm}, \mathbf{b}}^* : H^*(M; \Lambda_0) \rightarrow HF^*((L, \mathbf{b}), (L, \mathbf{b}); \Lambda_0) \quad (1.20)$$

is constructed in [FOOO1] Theorem 3.8.62. Utilizing this map $i_{\text{qm}, \mathbf{b}}^*$, we can locate μ_e^b -superheavy Lagrangian submanifolds in several circumstances.

Theorem 1.7. *Consider a pair (\mathbf{b}, \mathbf{b}) with $\mathbf{b} \in \mathcal{M}_{\text{weak,def}}(L; \Lambda_+)$ and $\pi_{\text{bulk}}(\mathbf{b}) = \mathbf{b}$. Let e be an idempotent of $QH_{\mathbf{b}}^*(M; \Lambda)$ such that*

$$i_{\text{qm}, \mathbf{b}}^*(e) \neq 0 \in HF^*((L, \mathbf{b}), (L, \mathbf{b}); \Lambda).$$

Then L is ζ_e^b -heavy and μ_e^b -heavy.

If e is a unit of a field factor of $QH_{\mathbf{b}}^(M; \Lambda)$ in addition, then L is ζ_e^b -superheavy and μ_e^b -superheavy.*

See Definition 18.5 for the definitions of μ_e^b -heavy and μ_e^b -superheavy sets. Theorem 1.7 (Theorem 18.8) is proved in Section 18.

Remark 1.8. (1) Theorem 1.7 gives rise to a proof of a conjecture made in [FOOO2] Remark 1.7.

(2) Theorem 1.7 is closely related to Theorem 1.20 [EP3].

Theorem 1.7 also proves linear independence of some spectral Calabi quasimorphisms in the following sense.

Definition 1.9. Let

$$\mu_j : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$$

be homogeneous Calabi-quasimorphisms for $j = 1, \dots, N$. We say that they are *linearly independent* if there exists a subgroup $\cong \mathbb{Z}^N$ of $\widetilde{\text{Ham}}(M, \omega)$ such that the restriction of $(\mu_1, \dots, \mu_N) : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}^N$ to this subgroup is an isomorphism to its lattice. A (possibly infinite) set of elements of $\widetilde{\text{Ham}}(M, \omega)$ is said to be *linearly independent* if any of its finite subset is linearly independent in the above sense. The case of $\text{Ham}(M, \omega)$ can be defined in the same way.

Corollary 1.10. *Let L_j be mutually disjoint relatively spin Lagrangian submanifolds. ($j = 1, \dots, N$.) Let $\mathbf{b}_j \in H^{\text{even}}(M; \Lambda_+)$ and $\mathbf{b}_j \in \mathcal{M}_{\text{weak, def}}(L_j; \Lambda_+)$ with $\pi_{\text{bulk}}(\mathbf{b}_j) = \mathbf{b}_j$. Let e_j be a unit of a field of factor of $QH_{\mathbf{b}_j}^*(M, \Lambda)$ such that*

$$i_{\text{qm}, \mathbf{b}_j}^*(e_j) \neq 0 \in HF^*((L_j, \mathbf{b}_j), (L_j, \mathbf{b}_j); \Lambda), \quad j = 1, \dots, N.$$

Then $\mu_{e_j}^{\mathbf{b}_j}$ ($j = 1, \dots, N$) are linearly independent.

This corollary follows from Theorem 1.7 mentioned above and [EP3] Theorem 8.2. (See also Section 19 of this paper.)

The study of toric manifolds [FOOO3] and deformations of some toric orbifolds [FOOO5] provides examples for which the hypothesis of Corollary 1.10 is satisfied. This study gives rise to the following theorem

Theorem 1.11. *Let M be one of the following three kinds of symplectic manifolds:*

- (1) $S^2 \times S^2$ with monotone toric symplectic structure,
- (2) Cubic surface,
- (3) k points blow up of $\mathbb{C}P^2$ with certain toric symplectic structure, where $k \geq 2$.

Then (M, ω) carries an uncountable set $\{\mu_a\}_{a \in \mathfrak{A}}$ of quasimorphisms

$$\mu_a : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$$

that are linearly independent.

Remark 1.12. (1) In the case of $(M, \omega) = S^2 \times S^2$, we have quasimorphisms

$$\mu_a : \text{Ham}(M, \omega) \rightarrow \mathbb{R} \text{ in place of } \mu_a : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}. \text{ See Corollary 23.6.}$$

- (2) We can explicitly specify the symplectic structure used in Theorem 1.11 (3). See Section 23.
- (3) We can also construct a similar example in higher dimension by the similar way.
- (4) Theorem 1.11 for $S^2 \times S^2$ was announced in [FOOO5] Remark 7.1, and for the case of k -points ($k \geq 2$) blow up of $\mathbb{C}P^2$ in [FOOO3] Remark 1.2 (3), respectively.
- (5) Biran-Entov-Polterovich constructed an uncountable family of linearly independent Calabi quasimorphisms for the case of the group $\text{Ham}(B^{2n}(1); \omega)$ of compactly supported Hamiltonian diffeomorphisms of balls with $n \geq 2$ in [BEP]. Theorem 1.11 provides the first example of closed M with such property.

- (6) For the case of $M = \mathbb{C}P^2$, existence of infinitely many homogeneous Calabi quasimorphisms on $\text{Ham}(M, \omega)$ is still an open problem.
- (7) Theorem 1.11 implies that the second bounded cohomology of $\widetilde{\text{Ham}}(M; \omega)$ is of infinite rank for (M, ω) appearing in Theorem 1.11. In fact, the *defect* Def_a defined by

$$\text{Def}_a(\phi, \psi) := \mu_a(\phi) + \mu_a(\psi) - \mu_a(\phi\psi)$$

defines a bounded two-cocycle. It follows from the simplicity of the group $\text{Ham}(M; \omega)$ [Ba1] that the set of cohomology classes of $\{\text{Def}_a\}$ is linearly independent in the 2nd bounded cohomology group of $\widetilde{\text{Ham}}(M, \omega)$.

- (8) In [FOOO9], we will study a generalization of Theorem 1.11 to a Kähler surface M , which is a smoothing of a toric orbifold with A_n -type singularities. See Section 24 for the A_2 -case.
- (9) At the final stage of completing this paper, a paper [Bor] appears in the arXiv which discusses a result related to Theorem 1.11 (3) using [AM].

Another corollary of Theorem 1.7 combined with Theorem 1.4 (iii), Theorem 1.8 [EP3] is the following intersection result of the exotic Lagrangian tori discovered in [FOOO5].

Theorem 1.13. *Let $T(u) \subset S^2(1) \times S^2(1)$ for $0 < u \leq 1/2$ be the tori from [FOOO5]. Then we have*

$$\psi(T(u)) \cap (S_{\text{eq}}^1 \times S_{\text{eq}}^1) \neq \emptyset$$

for any symplectic diffeomorphism ψ of $S^2(1) \times S^2(1)$.

Remark 1.14. Theorem 1.13 was announced in the introduction of [FOOO5]. The proof is given in Subsection 23.2.

A brief outline of the content of the paper is now in order. The present paper consist of 6 chapters. Chapter 1 is a review. In Chapter 2, we first enhance the Hamiltonian Floer theory by involving its deformations by ambient cohomology classes, which we call bulk deformations. In this paper, we use de Rham (co)cycles instead of singular cycles as in [FOOO3, FOOO6]. After this enhancement, we generalize construction of spectral invariants in [Oh4] involving bulk deformations and define *spectral invariants with bulk*. Chapter 3 then generalizes construction [EP2, EP1] of symplectic partial quasi-states and Calabi quasimorphisms by replacing the spectral invariants defined in [Oh4] by these spectral invariants with bulk.

In the course of carrying out these enhancements, we also unify, clarify and enhance many known constructions in Hamiltonian Floer theory in the framework of Kuranishi structures and accompanied abstract perturbation theory originally established in [FO] and further enhanced in Appendix A.2 of [FOOO1], [FOOO3, FOOO6, Fu3]. These are needed particularly because many constructions related to the study of spectral invariants (with bulk) have to be done in the chain level, not just in homology. Examples of such enhancement include construction of pants product [Sc1] and Piunikhin isomorphism whose construction was outlined in [Piu, RT, PSS]. We give complete construction of both of these in general compact symplectic manifolds without assuming any conditions on (M, ω) such as semi-positivity or rationality.

In Chapter 4, we connect the study of spectral invariants to the Lagrangian Floer theory developed in [FOOO1]. The main construction in the study is based on open-closed Gromov-Witten theory developed in [FOOO1] Section 3.8, which induces a map from the quantum cohomology of the ambient symplectic manifolds to the Hochschild cohomology of A_∞ algebra (or more generally that of Fukaya category of (M, ω)). A part of this map was also defined in [FOOO1] and further studied in [FOOO6] Section 31. This part borrows much from [FOOO1, FOOO6] in its exposition. The main new ingredient is a construction of a map from Floer homology of periodic Hamiltonians to Floer cohomology of Lagrangian submanifold, through which the map from quantum cohomology to Floer cohomology of Lagrangian submanifold factors (Subsection 18.4). We also study its properties especially those related to the filtration. A similar construction was used by Albers [Al] and also by Entov-Polterovich [EP3] in the monotone context.

In Chapter 5, we combine the results obtained in the previous chapters together with the results on the Lagrangian Floer theory of toric manifolds obtained in the series of our previous papers [FOOO2, FOOO3, FOOO5, FOOO6], give various new constructions of Calabi quasimorphisms and new Lagrangian intersections results on toric manifolds and other Kähler surfaces. These results are obtained by detecting the heavyness of Lagrangian submanifolds in the sense of Entov-Polterovich [EP3] in terms of spectral invariants, critical point theory of potential functions and also open-closed morphism between quantum cohomology to Hochschild cohomology of A_∞ -algebra of Lagrangian submanifolds.

Finally in Chapter 6, we prove various technical results necessary to complete the constructions carried out in the previous parts. For example, we establish the isomorphism property of the Piunikhin map with bulk. We give the construction of Seidel homomorphism with bulk extending the results of [Se] and generalize the McDuff-Tolman's representation of quantum cohomology ring of toric manifolds in terms of Seidel elements [MT] to that of big quantum cohomology ring.

We feel that the existing literature on the Hamiltonian Floer theory, spectral invariants and their applications do not contain many details on the transversality issue in the generality used in the present paper: Most of the literature assume semi-positivity but do not use Kuranishi structure and virtual cycle techniques developed in [FO] or do not give enough details of the latter virtual cycle techniques in their exposition. Moreover, various important lemmas and constructions related to Hamiltonian Floer theory and spectral invariants are scattered around here and there and sometimes with different conventions of Hamiltonian vector fields and the action functional in the literature. Because of these reasons, for readers's convenience and for the completeness' sake, we provide a fair amount of these details and proofs of those already in the literature in a unified and coherent fashion. We also provide those proofs in the most general context using the framework of Kuranishi structure and associated abstract perturbation theory, without imposing any restrictions on the ambient symplectic manifold (M, ω) .

Notations and Conventions

We follow the conventions of [Oh4, Oh6, Oh7] for the definition of Hamiltonian vector fields and action functional and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants and Entov-Polterovich's Calabi

quasimorphisms. There are differences from e.g., those used in [EP1, EP2, EP3] one way or the other. (See Remark 4.17 for the explaining the differences.)

- (1) The Hamiltonian vector field X_H is defined by $dH = \omega(X_H, \cdot)$.
- (2) The flow of X_H is denoted by $\phi_H : t \mapsto \phi_H^t$ and its time-one map by $\psi_H = \phi_H^1 \in \text{Ham}(M, \omega)$.
- (3) We denote by $[\phi_H]$ the path homotopy class of $\phi_H : [0, 1] \rightarrow \text{Ham}(M, \omega)$ relative to the ends which we generally denote $\psi_H = [\phi_H]$. We denote by $z_H^p(t) = \phi_H^t(p)$ the solution associated to a fixed point p of $\psi_H = \phi_H^1$.
- (4) $\tilde{H}(t, x) = -H(1-t, x)$ is the *time-reversal* Hamiltonian generating $\phi_H^{1-t} \phi_H^{-1}$.
- (5) We denote by $H_1 \# H_2$ the Hamiltonian generating the *concatenation* of the two Hamiltonian paths ϕ_{H_1} followed by $\phi_{H_2}^t$. More explicitly, it is defined by

$$(H_1 \# H_2)(t, x) = \begin{cases} 2H_1(2t, x) & 0 \leq t \leq 1/2 \\ 2H_2(2t-1, x) & 1/2 \leq t \leq 1. \end{cases}$$

(Warning: This notation is different from those used in [Oh4, Oh5, Oh6] where $(H_1 \# H_2)(t, x) = H_1(t, x) + H_2(t, (\phi_{H_1}^t)^{-1}(x))$ generating the product isotopy $t \mapsto \phi_{H_1}^t \phi_{H_2}^t$.)

- (6) The action functional $\mathcal{A}_H : \tilde{\mathcal{L}}_0(M) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int_0^1 H(t, \gamma(t)) dt.$$

- (7) \mathcal{J}_ω = the set of ω -compatible almost complex structures.
- (8) $j_\omega = \mathcal{L}(\mathcal{J}_\omega)$ = the set of S^1 -family J of compatible almost complex structures; $J = \{J_t\}_{t \in S^1}$.
- (9) $\mathcal{P}(j_\omega) = \text{Map}([0, 1] \times S^1, \mathcal{J}_\omega)$; $(s, t) \in [0, 1] \times S^1 \mapsto J_t^s \in \mathcal{J}_\omega$.
- (10) $\mathcal{K} = \{\chi : \mathbb{R} \rightarrow [0, 1]\}$ where χ is a smooth function with $\chi'(\tau) \geq 0$, $\chi(\tau) \equiv 0$ for $\tau \leq 0$ and $\chi(\tau) \equiv 1$ for $\tau \geq 1$. We define $\tilde{\chi}$ by $\tilde{\chi} = 1 - \chi$.
- (11) For given $H \in C^\infty([0, 1] \times S^1 \times M, \mathbb{R})$, we define the \mathbb{R} -family H_χ by

$$H_\chi(\tau, t, x) = \chi(\tau)H(t, x). \quad (1.21)$$

- (12) For $J \in \mathcal{P}(j_\omega)$ we take $J_s = \{J_{s,t}; t \in S^1\}$ such that

$$J_{1,t} = J_t, \quad J_{0,t} = J_0, \quad J_{s,0} = J_0,$$

and put

$$J_\chi(\tau, t) = J_{\chi(\tau), t}.$$

- (13) If $H \in C^\infty(S^1 \times M, \mathbb{R})$ and $J \in \mathcal{P}(j_\omega)$, we put

$$H^\chi(\tau, t, x) = H(\chi(\tau), t, x), \quad J^\chi(\tau, t, x) = J(\chi(\tau), t, x). \quad (1.22)$$

- (14) The *Piunikhin chain map*

$$\mathcal{P}_{(H_\chi, J_\chi)}^b : \Omega_*(M) \hat{\otimes} \Lambda^\downarrow \rightarrow CF_*(M, H; \Lambda^\downarrow)$$

is associated to (H_χ, J_χ) in (11), (12). (See Section 6). The map

$$\mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^b : CF_*(M, H; \Lambda^\downarrow) \rightarrow \Omega_*(M) \hat{\otimes} \Lambda^\downarrow$$

is associated to $\tilde{\chi}(\tau) = \chi(1 - \tau)$. (See Section 26.)

- (15) We denote the set of shuffles of ℓ elements by

$$\text{Shuff}(\ell) = \{(\mathbb{L}_1, \mathbb{L}_2) \mid \mathbb{L}_1 \cup \mathbb{L}_2 = \{1, \dots, \ell\}, \mathbb{L}_1 \cap \mathbb{L}_2 = \emptyset\}. \quad (1.23)$$

For $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$ let $\#\mathbb{L}_i$ be the order of this subset. Then $\#\mathbb{L}_1 + \#\mathbb{L}_2 = \ell$.

The set of *triple shuffles* is the set of $(\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3)$ such that $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3 = \{1, \dots, \ell\}$ and that $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$ are mutually disjoint.

- (16) The universal Novikov ring Λ_0 and its filed Λ of fractions are defined by

$$\begin{aligned} \Lambda_0 &= \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \rightarrow \infty} \lambda_i = +\infty \right\}, \\ \Lambda &= \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\} \cong \Lambda_0[T^{-1}]. \end{aligned}$$

The maximal ideal of Λ_0 is denoted by

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{>0}, \lim_{i \rightarrow \infty} \lambda_i = +\infty \right\}.$$

We define the valuation \mathbf{v}_T on Λ by

$$\mathbf{v}_T \left(\sum_{i=1}^{\infty} a_i T^{\lambda_i} \right) = \inf\{\lambda_i \mid a_i \neq 0\}, \quad \mathbf{v}_T(0) = +\infty.$$

We also use the following (downward) Novikov ring Λ_0^\downarrow and field Λ^\downarrow :

$$\begin{aligned} \Lambda_0^\downarrow &= \left\{ \sum_{i=1}^{\infty} a_i q^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\leq 0}, \lim_{i \rightarrow \infty} \lambda_i = -\infty \right\}, \\ \Lambda^\downarrow &= \left\{ \sum_{i=1}^{\infty} a_i q^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \lambda_i = -\infty \right\} \cong \Lambda_0^\downarrow[q]. \end{aligned}$$

The maximal ideal of Λ_0^\downarrow is denoted by

$$\Lambda_-^\downarrow = \left\{ \sum_{i=1}^{\infty} a_i q^{\lambda_i} \in \Lambda^\downarrow \mid \lambda_i < 0 \right\}.$$

We define the valuation \mathbf{v}_q on Λ^\downarrow by

$$\mathbf{v}_q \left(\sum_{i=1}^{\infty} a_i q^{\lambda_i} \right) = \sup\{\lambda_i \mid a_i \neq 0\}, \quad \mathbf{v}_q(0) = -\infty.$$

Of course, Λ_0^\downarrow and Λ^\downarrow are isomorphic to Λ_0 and Λ respectively by the isomorphism $q \mapsto T^{-1}$. Under the isomorphism we have $\mathbf{v}_q = -\mathbf{v}_T$. The downward universal Novikov rings seem to be more commonly used in the study of spectral invariant (e.g., [Oh4]), while the upward versions Λ and Λ_0 are used in Lagrangian Floer theory (e.g., [FOOO1]).

- (17) Sometimes we regard the de Rham complex $(\Omega(M), d)$ as a chain complex and consider its homology. In that case we put

$$\Omega_k(M) = \Omega^{\dim M - k}(M), \quad \partial = (-1)^{\deg + 1} d.$$

See Remark 3.5.8 [FOOO1] for this sign convention. When a cohomology class $a \in H^{\dim M - k}(M)$ is represented by a differential form α and we regard α as an element of the chain complex $(\Omega_*(M), \partial)$, we denote the homology class by $a^\flat \in H_k(M)$.

- (18) When we say that the boundary orientation of some moduli space is compatible with the orientation of strata corresponding to bubbling off disks (with boundary marked points), the compatibility means in the sense of Proposition 8.3.3 in [FOOO1].
- (19) Let V be a \mathbb{Z} graded vector space over \mathbb{C} . We put $B_k V = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$

and $BV = \bigoplus_{k=0}^{\infty} B_k V$ where $B_0 V = \mathbb{C}$. Then BV has a structure of coassociative coalgebra with coproduct. We note that we have two kinds of coproduct structures on BV . One is the *deconcatenation coproduct* defined by

$$\Delta_{\text{decon}}(x_1 \otimes \cdots \otimes x_k) = \sum_{i=0}^k (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k). \quad (1.24)$$

The other is the *shuffful coproduct* defined by

$$\begin{aligned} & \Delta_{\text{shuff}}(x_1 \otimes \cdots \otimes x_k) \\ &= \sum_{(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(k)} (-1)^*(x_{\ell_1(1)} \otimes \cdots \otimes x_{\ell_1(k_1)}) \otimes (x_{\ell_2(1)} \otimes \cdots \otimes x_{\ell_2(k_2)}), \end{aligned} \quad (1.25)$$

where $\mathbb{L}_j = \{\ell_j(1), \dots, \ell_j(k_j)\}$ with $\ell_j(1) < \cdots < \ell_j(k_j)$ for $j = 1, 2$ and

$$* = \sum_{\ell_1(i) > \ell_2(j)} \deg x_{\ell_1(i)} \deg x_{\ell_2(j)}. \quad (1.26)$$

It is easy to see that

$$\begin{aligned} \Delta_{\text{decon}} \left(\sum_{k=0}^{\infty} x^{\otimes k} \right) &= \left(\sum_{k=0}^{\infty} x^{\otimes k} \right) \otimes \left(\sum_{k=0}^{\infty} x^{\otimes k} \right) \\ \Delta_{\text{shuff}} \left(\sum_{k=0}^{\infty} \frac{x^{\otimes k}}{k!} \right) &= \left(\sum_{k=0}^{\infty} \frac{x^{\otimes k}}{k!} \right) \otimes \left(\sum_{k=0}^{\infty} \frac{x^{\otimes k}}{k!} \right) \end{aligned} \quad (1.27)$$

if $\deg x$ is even. We write $e^x = \sum_{k=0}^{\infty} x^{\otimes k}$ or $e^x = \sum_{k=0}^{\infty} \frac{x^{\otimes k}}{k!}$ according as we use Δ_{decon} or Δ_{shuff} as coproduct structures.

- (20) The symmetric group \mathfrak{S}_k of order $k!$ acts on $B_k V$ by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) = (-1)^* x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where $*$ = $\sum_{i < j; \sigma(i) > \sigma(j)} \deg x_i \deg x_j$. We denote by $E_k V$ the quotient of $B_k V$ by the submodule generated by $\sigma \cdot \mathbf{x} - \mathbf{x}$ for $\sigma \in \mathfrak{S}_k$, $\mathbf{x} \in B_k V$. We denote by $[\mathbf{x}]$ an element of $E_k V$ and put $EV = \bigoplus_{k=0}^{\infty} E_k V$. The shuffle coproduct structure on BV induces a coproduct structure on EV , which we also denote by Δ_{shuff} . It is given by

$$\begin{aligned} & \Delta_{\text{shuff}}([x_1 \otimes \cdots \otimes x_k]) \\ &= \sum_{(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(k)} (-1)^* ([x_{\ell_1(1)} \otimes \cdots \otimes x_{\ell_1(k_1)}]) \otimes ([x_{\ell_2(1)} \otimes \cdots \otimes x_{\ell_2(k_2)}]). \end{aligned} \quad (1.28)$$

Here $*$ is the same as (1.26). Then EV becomes a coassociative and graded cocommutative coalgebra.

In [FOOO1], [FOOO2], [FOOO3], we denote by $E_k V$ the \mathfrak{S}_k -invariant subset of $B_k V$ and use the deconcatenation coproduct restricted to the subset. In [FOOO6], we use $E_k V$ as the *quotient space* and the shuffle coproduct on it as we do in this paper. This paper follows the conventions used in [FOOO6].

(21) Let L be a relatively spin closed Lagrangian submanifold of a symplectic manifold (M, ω) .

(a) For the case $V = \Omega(L)[1]$, we always use the deconcatenation coproduct Δ_{decon} on $B(\Omega(L)[1])$.

(b) For the case $V = \Omega(M)[2]$, we always use the shuffle coproduct Δ_{shuff} on $E(\Omega(M)[2])$.

Here $\Omega(L)[1]$ (resp. $\Omega(M)[2]$) is the degree shift by $+1$ of $\Omega(L)$, i.e., $(\Omega(L)[1])^d = \Omega^{d+1}(L)$ (resp. $+2$ of $\Omega(M)$, i.e., $(\Omega(M)[2])^d = \Omega^{d+2}(M)$.) Therefore, no confusion can occur even if we use the same notation Δ for the coproducts Δ_{decon} and Δ_{shuff} .

Part 1. Review of spectral invariants

2. HAMILTONIAN FLOER-NOVIKOV COMPLEX

Let $\tilde{\mathcal{L}}_0(M)$ be the set of all the pairs $[\gamma, w]$ where γ is a loop $\gamma : S^1 \rightarrow M$ and $w : D^2 \rightarrow M$ a disc with $w|_{\partial D^2} = \gamma$. We identify $[\gamma, w]$ and $[\gamma', w']$ if $\gamma = \gamma'$ and w is homotopic to w' relative to the boundary γ . When a one-periodic Hamiltonian $H : (\mathbb{R}/\mathbb{Z}) \times M \rightarrow \mathbb{R}$ is given, we consider the perturbed functional $\mathcal{A}_H : \tilde{\mathcal{L}}_0(M) \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int H(t, \gamma(t)) dt. \quad (2.1)$$

For a Hamiltonian $H : [0, 1] \times M \rightarrow \mathbb{R}$, we denote its flow, a Hamiltonian isotopy, by $\phi_H : t \mapsto \phi_H^t \in \text{Ham}(M, \omega)$. This gives a one-to-one correspondence between equivalence classes of H modulo the addition of a function on $[0, 1]$ and Hamiltonian isotopies. We denote the time-one map by $\psi_H := \phi_H^1$. We put

$$\text{Fix } \psi_H = \{p \in M \mid \psi_H(p) = p\}.$$

Each element $p \in \text{Fix } \psi_H$ induces a map $z_p = z_p^H : S^1 \rightarrow M$ by the correspondence

$$z_p(t) = \phi_H^t(p), \quad (2.2)$$

where $t \in \mathbb{R}/\mathbb{Z} \cong S^1$. The loop z_p satisfies Hamilton's equation

$$\dot{x} = X_H(t, x).$$

Here X_H is the (time-dependent) Hamiltonian vector field given by $X_H(t, x) = X_{H_t}(x)$ where X_{H_t} is the Hamiltonian vector field generated by the function $H_t : C^\infty(M) \rightarrow \mathbb{R}$. We denote by $\text{Per}(H)$ the set of one-periodic solutions of $\dot{x} = X_H(t, x)$. Then (2.2) provides a one-to-one correspondence between $\text{Fix } \psi_H$ and $\text{Per}(H)$. The next lemma is well-known.

Lemma 2.1. *The set of critical points of \mathcal{A}_H is given by*

$$\text{Crit}(\mathcal{A}_H) = \{[\gamma, w] \mid \gamma \in \text{Per}(H), w|_{\partial D^2} = \gamma\}.$$

Hereafter we assume that our Hamiltonian H is normalized in the sense of (1.5) unless otherwise stated explicitly.

The Floer homology theory [Fl] of periodic Hamiltonian system is the semi-infinite version of the Morse theory of the function \mathcal{A}_H on an appropriate covering space of the space $\mathcal{L}_0(M)$ of contractible loops.

We say that H or its associated map ψ_H is *non-degenerate* if at $p \in \text{Fix } \psi_H$, the differential $d_p \psi_H : T_p M \rightarrow T_p M$ does not have eigenvalue 1. The cardinality of $\text{Per}(H)$ is finite if ψ_H is nondegenerate.

We recall from Notations and Conventions (16) in Section 1 that we define a valuation \mathfrak{v}_q on the (downward) universal Novikov field Λ^\downarrow by

$$\mathfrak{v}_q \left(\sum_{i=1}^{\infty} a_i q^{\lambda_i} \right) = \sup \{ \lambda_i \mid a_i \neq 0 \}. \quad (2.3)$$

It satisfies the following properties:

- (1) $\mathfrak{v}_q(xy) = \mathfrak{v}_q(x) + \mathfrak{v}_q(y)$,
- (2) $\mathfrak{v}_q(x + y) \leq \max\{\mathfrak{v}_q(x), \mathfrak{v}_q(y)\}$,
- (3) $\mathfrak{v}_q(x) = -\infty$ if and only if $x = 0$,
- (4) $\mathfrak{v}_q(q) = 1$,

$$(5) \quad \mathfrak{v}_q(ax) = \mathfrak{v}_q(x) \text{ if } a \in \mathbb{C} \setminus \{0\}.$$

We consider the Λ^\perp vector space $\widehat{CF}(M; H; \Lambda^\perp)$ with basis given by the critical point set $\text{Crit}(\mathcal{A}_H)$ of \mathcal{A}_H .

Definition 2.2. We define an equivalence relation \sim on $\widehat{CF}(M; H; \Lambda^\perp)$ so that $[\gamma, w] \sim q^c[\gamma', w']$ if and only if

$$\gamma = \gamma', \quad \int_{D^2} w'^* \omega = \int_{D^2} w^* \omega - c. \quad (2.4)$$

The quotient of $\widehat{CF}(M; H; \Lambda^\perp)$ modded out by this equivalence relation \sim is called the Floer complex of the periodic Hamiltonian H and denoted by $CF(M; H; \Lambda^\perp)$.

Here we do not assume the condition on the Conley-Zehnder indices and work with \mathbb{Z}_2 -grading.

In the literature on Hamiltonian Floer homology, additional requirement

$$c_1(\overline{w} \# w') = 0$$

is imposed in the definition of Floer complex, denoted by $CF(H)$. For the purpose of the current paper, the equivalence relation (2.4) is enough and more favorable in that it makes the associated Novikov ring becomes a field. To differentiate the current definition from $CF(H)$, we denote the complex used in the present paper by $CF(M, H)$ or $CF(M, H; \Lambda^\perp)$.

Lemma 2.3. *As a Λ^\perp vector space, $CF(M, H; \Lambda^\perp)$ is isomorphic to the direct sum $\Lambda \# \text{Per}(H)$.*

Moreover the following holds: We fix a lifting $[\gamma, w_\gamma] \in \text{Crit}(\mathcal{A}_H)$ for each $\gamma \in \text{Per}(H)$. Then any element x of $CF(M, H; \Lambda^\perp)$ is uniquely written as a sum

$$x = \sum_{\gamma \in \text{Per}(H)} x_\gamma [\gamma, w_\gamma], \quad \text{with } x_\gamma \in \Lambda^\perp. \quad (2.5)$$

The proof is easy and omitted.

Definition 2.4. (1) Let x be as in (2.5). We define

$$\mathfrak{v}_q(x) = \max\{\mathfrak{v}_q(x_\gamma) + \mathcal{A}_H([\gamma, w_\gamma]) \mid \gamma \in \text{Per}(H)\}.$$

(2) We define a filtration $F^\lambda CF(M, H; \Lambda^\perp)$ on $CF(M, H; \Lambda^\perp)$ by

$$F^\lambda CF(M, H; \Lambda^\perp) = \{x \in CF(M, H; \Lambda^\perp) \mid \mathfrak{v}_q(x) \leq \lambda\}.$$

We have

$$F^{\lambda_1} CF(M, H; \Lambda^\perp) \subset F^{\lambda_2} CF(M, H; \Lambda^\perp)$$

if $\lambda_1 < \lambda_2$. We also have

$$\bigcap_{\lambda} F^\lambda CF(M, H; \Lambda^\perp) = \{0\}, \quad \bigcup_{\lambda} F^\lambda CF(M, H; \Lambda^\perp) = CF(M, H).$$

(3) We define a metric d_q on $CF(M, H; \Lambda^\perp)$ by

$$d_q(x, x') = e^{\mathfrak{v}_q(x - x')}. \quad (2.6)$$

(2.3), (2.4) and Definition 2.4 imply that

$$\mathfrak{v}_q(a\mathfrak{x}) = \mathfrak{v}_q(a) + \mathfrak{v}_q(\mathfrak{x})$$

for $a \in \Lambda^\downarrow$, $\mathfrak{x} \in CF(M, H; \Lambda^\downarrow)$. We also have

$$q^{\lambda_1} F^{\lambda_2} CF(M, H; \Lambda^\downarrow) \subseteq F^{\lambda_1 + \lambda_2} CF(M, H; \Lambda^\downarrow).$$

Lemma 2.5. (1) \mathfrak{v}_q is independent of the choice of the lifting $\gamma \mapsto [\gamma, w_\gamma]$.
 (2) $CF(M; H; \Lambda^\downarrow)$ is complete with respect to the metric d_q .
 (3) The infinite sum

$$\sum_{[\gamma, w] \in \text{Crit } \mathcal{A}_H} x_{[\gamma, w]}[\gamma, w]$$

converges in $CF(M; H; \Lambda^\downarrow)$ with respect to the metric d_q if

$$\{[\gamma, w] \in \text{Crit } \mathcal{A}_H \mid \mathfrak{v}_q(x_{[\gamma, w]}) + \mathcal{A}_H([\gamma, w]) > -C, x_{[\gamma, w]} \neq 0\}.$$

is finite for any $C \in \mathbb{R}$.

The proof is easy and omitted.

3. FLOER BOUNDARY MAP

In this section we define the boundary operator $\partial_{(H, J)}$ on $CF(M; H; \Lambda^\downarrow)$ so that it becomes a filtered complex. Suppose H is a non-degenerate one-periodic Hamiltonian function and a one-periodic $J = \{J_t\}_{t \in S^1}$ of compatible almost complex structures. The study of the following perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0 \quad (3.1)$$

is the heart of the Hamiltonian Floer theory. Here and hereafter J in (3.1) means J_t .

Remark 3.1. (1) In this paper, we *never* use perturbation of (a family of) compatible almost complex structures J to achieve transversality of the moduli space of the Floer equations (3.1) but use abstract perturbations (multisections of the Kuranishi structure) to achieve necessary transversality.
 (2) In Chapters 1-3 (and somewhere in the appendix) we use a $t \in S^1$ parametrized family of compatible almost complex structures $\{J_t\}_t$. However we emphasize that we do *not* need to use a $t \in S^1$ parametrized family of compatible almost complex structures but can use a *fixed* compatible almost complex structure J , to prove all of our main results of this paper. (We need to use t -dependent J for the construction in Sections 29 and 30.) The t -dependent J is included only for the sake of consistency with the reference on spectral invariants. (Traditionally t -dependent J had been used to achieve transversality. As we mentioned in (1), we do *not* need this extra freedom in this paper since we use abstract perturbations.)

The following definition is useful for the later discussions.

Definition 3.2. Let $\gamma, \gamma' \in \text{Per}(H)$. We denote by $\pi_2(\gamma, \gamma')$ the set of homotopy classes of smooth maps $u : [0, 1] \times S^1 \rightarrow M$ relative to the boundary $u(0, t) = \gamma(t)$, $u(1, t) = \gamma'(t)$. We denote by $[u] \in \pi_2(\gamma, \gamma')$ its homotopy class.

We define by $\pi_2(\gamma)$ the set of relative homotopy classes of the maps $w : D^2 \rightarrow M; w|_{\partial D^2} = \gamma$. For $C \in \pi_2(\gamma, \gamma')$, there is a natural map of $(\cdot) \# C : \pi_2(\gamma) \rightarrow \pi_2(\gamma')$ induced by the gluing map $w \mapsto w \# C$. There is also the natural gluing map

$$\pi_2(\gamma_0, \gamma_1) \times \pi_2(\gamma_1, \gamma_2) \rightarrow \pi_2(\gamma_0, \gamma_2), \quad (u_1, u_2) \mapsto u_1 \# u_2.$$

For each $[\gamma, w], [\gamma', w'] \in \text{Crit}(\mathcal{A}_H)$, we will define a moduli space

$$\mathcal{M}(H, J; [\gamma, w], [\gamma', w']).$$

We begin with the definition of the energy.

Definition 3.3. (Energy) For a given smooth map $u : \mathbb{R} \times S^1 \rightarrow M$, we define the energy of u by

$$E_{(H, J)}(u) = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_J^2 + \left| \frac{\partial u}{\partial t} - X_{H_t}(u) \right|_J^2 \right) dt d\tau.$$

Definition 3.4. We denote by $\widehat{\mathcal{M}}(H, J; [\gamma, w], [\gamma', w'])$ the set of all maps $u : \mathbb{R} \times S^1 \rightarrow M$ which satisfy the following conditions:

- (1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0. \quad (3.2)$$

- (2) The energy $E_{(H, J)}(u)$ is finite.

- (3) The map u satisfies the following asymptotic boundary condition.

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = \gamma(t), \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = \gamma'(t).$$

- (4) The concatenation $w \# u$ of w and u is homotopic to w' .

It has an \mathbb{R} -action of translations in τ -direction. We denote the quotient space of this \mathbb{R} -action by $\mathring{\mathcal{M}}(H, J; [\gamma, w], [\gamma', w'])$.

When $[\gamma, w] = [\gamma', w']$, we set the space $\mathring{\mathcal{M}}(H, J; [\gamma, w], [\gamma, w])$ to be the empty set by definition.

Remark 3.5. The conditions (1) and (2) above make the convergence in (3) one of an exponential order, which in turn enables the statement (4) to make sense.

Denote by $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ the Conley-Zehnder index [CZ].

Proposition 3.6. (1) *The moduli space $\mathring{\mathcal{M}}(H, J; [\gamma, w], [\gamma', w'])$ has a compactification $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ that is Hausdorff.*

- (2) *The space $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ has an orientable Kuranishi structure with corners.*

- (3) *The boundary of $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ in the sense of Kuranishi structure is described by*

$$\begin{aligned} & \partial \mathcal{M}(H, J; [\gamma, w], [\gamma', w']) \\ &= \bigcup \mathcal{M}(H, J; [\gamma, w], [\gamma'', w'']) \times \mathcal{M}(H, J; [\gamma'', w''], [\gamma', w']), \end{aligned} \quad (3.3)$$

where the union is taken over all $[\gamma'', w''] \in \text{Crit}(\mathcal{A}_H)$.

- (4) *There exists a map $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$, Conley-Zehnder index, such that the (virtual) dimension satisfies the following equality (3.4).*

$$\dim \mathcal{M}(H, J; [\gamma, w], [\gamma', w']) = \mu_H([\gamma', w']) - \mu_H([\gamma, w]) - 1. \quad (3.4)$$

- (5) We can define orientations of $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ so that (3) above is compatible with this orientation.

This is proved in [FO] Chapter 4. More precisely, (1) is [FO] Theorem 19.12, (2), (3), (4) are [FO] Theorem 19.14 and (5) is [FO] Lemma 21.4. See [Fl, HS, On] etc. for the earlier works for the semi-positive cases.

We use the Conley-Zehnder index μ_H of $[\gamma, w]$ to define a \mathbb{Z}_2 grading on the Λ^\downarrow -vector space $CF(M; H)$. Namely, the homological degree of $[\gamma, w]$ is defined to be $n - \mu_H([\gamma, w])$. We remark that if $[w], [w'] \in \pi_2(\gamma)$ we have

$$\mu_H([\gamma, w']) - \mu_H([\gamma, w]) = -2c_1(M) \cap [\overline{w} \# w']$$

where $\overline{w} \# w'$ is a 2-sphere obtained by gluing \overline{w} and w' along γ where \overline{w} is the w with opposite orientation. (See [Fl] page 557.) In particular, it implies that the parity of $\mu_H([\gamma, w])$ depends only on $\gamma \in \text{Per}(H)$ but not on its lifting $[\gamma, w] \in \text{Crit } \mathcal{A}_H$

$$\begin{aligned} CF_1(M, H; \Lambda^\downarrow) &= \bigoplus_{\gamma; \mu_H([\gamma, w_\gamma]) + n \text{ is odd.}} \Lambda^\downarrow[\gamma, w_\gamma], \\ CF_0(M, H; \Lambda^\downarrow) &= \bigoplus_{\gamma; \mu_H([\gamma, w_\gamma]) + n \text{ is even.}} \Lambda^\downarrow[\gamma, w_\gamma]. \end{aligned} \quad (3.5)$$

Remark 3.7. We remark that the degree of Floer chain defined above is shifted by n from Conley-Zehnder index μ_H . By this shift, the degree will coincide with the degree of (quantum) cohomology group of M by the isomorphism in Theorem 3.10. See also Remark 5.5 (2).

We use Proposition 3.6 to define the Floer boundary map

$$\partial_{(H, J)} : CF_{k+1}(M, H; \Lambda^\downarrow) \rightarrow CF_k(M, H; \Lambda^\downarrow)$$

as follows.

We construct a system of multisections \mathfrak{s} on $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ inductively over the symplectic area $(\overline{w} \# w') \cap \omega \in \mathbb{R}_{\geq 0}$ which are transversal to 0 and compatible with the identification made in Proposition 3.6 (3). Such an inductive construction is proven to be possible for the relative version of the construction of Kuranishi structures in [FO] Theorem 6.12 (that is, [FOOO1] Lemma A1.20). Now we define

$$\partial_{(H, J)}[\gamma, w] = \sum_{[\gamma', w']} \# \mathcal{M}(H, J; [\gamma, w], [\gamma', w'])^{\mathfrak{s}} [\gamma', w']. \quad (3.6)$$

Here the sum is taken over all $[\gamma', w']$ satisfying $\mu_H([\gamma, w]) - \mu_H([\gamma', w']) = 1$. The rational number $\# \mathcal{M}(H, J; [\gamma, w], [\gamma', w'])^{\mathfrak{s}}$ is the virtual fundamental 0-chain of $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ with respect to the multisection \mathfrak{s} . Namely it is the order of the zero set of \mathfrak{s} counted with sign and multiplicity. (See [FO] Definition 4.6 or [FOOO1] Definition A1.28 for its precise definition.) Hereafter we omit \mathfrak{s} and simply write $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ for the perturbed moduli space.

By the Gromov-Floer compactness, the set of all $[\gamma', w'] \in \text{Crit } \mathcal{A}_H$ satisfying

$$\mathcal{M}(H, J; [\gamma, w], [\gamma', w']) \neq \emptyset, \quad [\overline{w} \# w'] \cap [\omega] < A$$

is finite, for any fixed $A \in \mathbb{R}$. Therefore Lemma 2.5 (3) implies that the right hand side of (3.6) converges in d_q -metric. We can prove

$$\partial_{(H, J)} \circ \partial_{(H, J)} = 0 \quad (3.7)$$

by applying Proposition 3.6 (3) in the case when $\mu_H([\gamma, w]) - \mu_H([\gamma', w']) = 2$. (See [FO] Lemma 20.2.)

Lemma 3.8. *For any $\lambda \in \mathbb{R}$,*

$$\partial_{(H,J)}(F^\lambda CF(M, H; \Lambda^\downarrow)) \subset F^\lambda CF(M, H; \Lambda^\downarrow).$$

Proof. If $u \in \widetilde{\mathcal{M}}(H, J; [\gamma, w], [\gamma', w'])$ then

$$\begin{aligned} \int u^* \omega &= \int_{\tau \in \mathbb{R}} \int_{t \in S^1} \omega \left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t} \right) dt d\tau \\ &= \int_{\tau \in \mathbb{R}} \int_{t \in S^1} \omega \left(\frac{\partial u}{\partial \tau}, J \frac{\partial u}{\partial \tau} + X_{H_t}(u) \right) dt d\tau \\ &= \int_{\tau \in \mathbb{R}} \int_{t \in S^1} \left| \frac{\partial u}{\partial \tau} \right|_J^2 dt d\tau - \int_{\tau \in \mathbb{R}} \int_{t \in S^1} (dH_t(u(\tau, t))) \left(\frac{\partial u}{\partial \tau} \right) dt d\tau \\ &= \int_{\tau \in \mathbb{R}} \int_{t \in S^1} \left| \frac{\partial u}{\partial \tau} \right|_J^2 dt d\tau - \int_{\tau \in \mathbb{R}} \left(\int_{t \in S^1} \frac{\partial}{\partial \tau} (H_t(u(\tau, t))) dt \right) d\tau \\ &= E_{(H,J)}(u) - \int_{t \in S^1} H_t(\gamma'(t)) dt + \int_{t \in S^1} H_t(\gamma(t)) dt. \end{aligned} \quad (3.8)$$

Therefore

$$\int u^* \omega + \int_{t \in S^1} H_t(\gamma'(t)) dt - \int_{t \in S^1} H_t(\gamma(t)) dt = E_{(H,J)}(u) \geq 0.$$

Combined with $w \# u \sim w'$, this implies

$$\mathcal{A}_H([\gamma', w']) = \mathcal{A}_H([\gamma, w]) - E_{(H,J)}(u) \leq \mathcal{A}_H([\gamma, w])$$

and hence Lemma 3.8 holds. \square

Definition 3.9. The Floer homology with Λ^\downarrow coefficients is defined by

$$HF_*(H, J; \Lambda^\downarrow) := \frac{\text{Ker } \partial_{(H,J)}}{\text{Im } \partial_{(H,J)}}.$$

Theorem 3.10. *We may choose the orientation in Proposition 3.6 (5) so that $HF_*(H, J; \Lambda^\downarrow)$ is isomorphic to the singular (co)homology $H(M; \Lambda^\downarrow)$ with Λ^\downarrow coefficients.*

This is proved in [FO] Theorem 22.1. We will describe a construction of isomorphism (which is different from the one in [FO]) below because we need to specify the isomorphism to encode each spectral invariant by the corresponding quantum cohomology class.

Definition 3.11. Consider a smooth function $\chi : \mathbb{R} \rightarrow [0, 1]$ with the properties

$$\chi(\tau) = \begin{cases} 0 & \text{for } \tau \leq 0 \\ 1 & \text{for } \tau \geq 1 \end{cases} \quad (3.9)$$

$$\chi'(\tau) \geq 0. \quad (3.10)$$

We denote by \mathcal{K} the set of such elongation functions.

We note that \mathcal{K} is convex and so contractible.

For given S^1 -dependent family of almost complex structures J , we consider the 2-parameter family $\{J_s\}_{s \in [0,1]}$ such that

$$J_{0,t} = J_0, \quad J_{1,t} = J_t \quad (3.11)$$

where J_0 is a time-independent almost complex structure. We also assume

$$J_{s,t} \equiv J_0, \quad \text{for } (s,t) \in \partial[0,1]^2 \setminus (\{1\} \times [0,1]).$$

For each nondegenerate $H : S^1 \times M \rightarrow \mathbb{R}$ we define $(\mathbb{R} \times S^1)$ -family to (H_χ, J_χ) on $\mathbb{R} \times S^1$ by

$$H_\chi(\tau, t) = \chi(\tau)H_t, \quad J_\chi(\tau, t) = J_{\chi(\tau),t}. \quad (3.12)$$

Remark 3.12. It is very important that the family J_s is t -independent for $s = 0$. See the proof of Proposition 6.11.

Definition 3.13. We denote by $\mathring{\mathcal{M}}(H_\chi, J_\chi; *, [\gamma; w])$ the set of all maps $u : \mathbb{R} \times S^1 \rightarrow M$ satisfying the following conditions:

- (1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J_\chi \left(\frac{\partial u}{\partial t} - \chi(\tau)X_{H_t}(u) \right) = 0. \quad (3.13)$$

Here and hereafter J_χ in (3.13) means $J_{\chi,t}$.

- (2) The energy

$$E_{(H_\chi, J_\chi)}(u) := \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_\chi}^2 + \left| \frac{\partial u}{\partial t} - \chi(\tau)X_{H_t}(u) \right|_{J_\chi}^2 \right) dt d\tau$$

is finite.

- (3) The map u satisfies the following asymptotic boundary condition:

$$\lim_{\tau \rightarrow +\infty} u(\tau, t) = \gamma(t).$$

- (4) The homotopy class of $[u] = [w]$ in $\pi_2(\gamma)$.

We note that since $\chi(\tau)X_{H_t} \equiv 0$ and $J_{\chi(\tau)} \equiv J_0$ for $\tau < -1$, which turns (3.13) into the genuine J_0 -holomorphic curve equation, the removable singularity theorem (due to Sacks-Uhlenbeck and Gromov, see e.g., [Si] Theorem 4.5.1) gives rise to a well-defined limit

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) \quad (3.14)$$

which does not depend on t . Therefore the homotopy class condition required in (4) above makes sense.

We denote this assignment of the limit by

$$\text{ev}_{-\infty} : \mathcal{M}(H_\chi, J_\chi; *, [\gamma, w]) \rightarrow M. \quad (3.15)$$

Here $*$ stands for a point in M which is the limit at $\tau = -\infty$ of the element in $\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])$.

Proposition 3.14. (1) The moduli space $\mathring{\mathcal{M}}(H_\chi, J_\chi; *, [\gamma, w])$ has a compactification $\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])$ that is Hausdorff.

- (2) The space $\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])$ has an orientable Kuranishi structure with corners.

(3) The boundary of $\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])$ is described by

$$\begin{aligned} & \partial \mathcal{M}(H_\chi, J_\chi; *, [\gamma, w]) \\ &= \bigcup \mathcal{M}(H_\chi, J_\chi; *, [\gamma', w']) \times \mathcal{M}(H, J; , [\gamma', w'], [\gamma, w]), \end{aligned} \quad (3.16)$$

where the union is taken over all $[\gamma', w'] \in \text{Crit}(\mathcal{A}_H)$.

(4) The (virtual) dimension satisfies the following equality (3.17).

$$\dim \mathcal{M}(H_\chi, J_\chi; *, [\gamma, w]) = \mu_H([\gamma, w]) + n. \quad (3.17)$$

(5) We can define a system of orientations of $\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])$ so that (3) above is compatible with this orientation.

(6) The map $\text{ev}_{-\infty}$ becomes a weakly submersive map in the sense of [FOOO1] Definition A1.13.

The proof is the same as that of Proposition 3.6 and is omitted.

We take a system of multisections \mathfrak{s} on $\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])$ for various $[\gamma, w]$ so that it is compatible at the boundary described in (3.16).

Let h be a differential k form on M . We define

$$\mathcal{P}_{(H_\chi, J_\chi)}(h) = \sum_{[\gamma, w]} \left(\int_{\mathcal{M}(H_\chi, J_\chi; *, [\gamma, w])^\mathfrak{s}} \text{ev}_{-\infty}^*(h) \right) [\gamma, w]. \quad (3.18)$$

(The symbol \mathcal{P} stands for Piunikhin [Piu].) Here the sum is taken over $[\gamma, w]$ with $\mu_H([\gamma, w]) = k - n$. The integration over the zero set of the multisection of the Kuranishi structure is defined for example in [FOOO2] Appendix C. By Gromov-Floer compactness we can prove that the right hand side is an element of $CF(M, H; \Lambda^\downarrow)$.

The definition (3.18) induces a map

$$\mathcal{P}_{(H_\chi, J_\chi)} : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow CF(M, H; \Lambda^\downarrow).$$

Here $\widehat{\otimes}$ is the completion of the algebraic tensor product (over \mathbb{R}) with respect to the norm \mathfrak{v}_q .

Let $(\Omega^*(M), d)$ be the de Rham complex of M . We regard it as a *chain* complex $(\Omega_*(M), \partial)$, where

$$\Omega_k(M) = \Omega^{\dim M - k}(M), \quad \partial = (-1)^{\deg + 1} d. \quad (3.19)$$

Lemma 3.15. $\mathcal{P}_{(H_\chi, J_\chi)}$ defines a chain map

$$\mathcal{P}_{(H_\chi, J_\chi)} : (\Omega(M), \partial) \widehat{\otimes} \Lambda^\downarrow \rightarrow (CF(M, H; \Lambda^\downarrow), \partial_{(H, J)})$$

from the de Rham complex to the Floer complex.

Proof. We can prove $\mathcal{P}_{(H_\chi, J_\chi)} \circ \partial = \partial_{(H, J)} \circ \mathcal{P}_{(H_\chi, J_\chi)}$ by Stokes' theorem ([FOOO2] Lemma C.9), Proposition 3.14 (3) and the definition. \square

Lemma 3.16. $\mathcal{P}_{(H_\chi, J_\chi)}$ induces a chain homotopy equivalence.

The proof is similar to the argument established in various similar situations. (One of the closest descriptions we can find in the literature is [FOOO3] Section 8 Proposition 8.24, where a similar lemma is proved in the case of Lagrangian Floer theory.) We give a proof in the appendix for completeness' sake.

- Remark 3.17.** (1) Actually there is a problem of running-out as mentioned in [FOOO1] Section 7.2.3. In order to handle it we first work over Λ_0^\downarrow coefficients and stop the construction at some energy level. Then we take an inductive limit. The technical difficulty to perform this construction is much simpler than that of [FOOO1] (Section 7.2), since we here need to take an inductive limit of chain complex (or DGA) which is much simpler than A_∞ algebra in general (which is discussed in [FOOO1] Section 7.2). So we omit the detail.
- (2) Here we work over Λ^\downarrow coefficients and so with \mathbb{Z}_2 grading given in (3.5). Under the assumption that minimal Chern number is $2N$, we can define a \mathbb{Z}_{2N} grading.
- (3) Here we use \mathbb{C} as the ground field. Up until now, we can work with \mathbb{Q} in the same way. We prefer to use \mathbb{C} since we will use de Rham theory later on to involve bulk deformations in our constructions. In addition, the de Rham theory is used for the Lagrangian Floer theory of toric manifolds in various calculations and applications developed in [FOOO2] etc.
- (4) We use de Rham cohomology of M to define $\mathcal{P}_{(H_\chi, J_\chi)}$. There are several other ways of constructing this isomorphism. One uses singular (co)homology as in [FOOO1] Section 7.2 (especially Proposition 7.2.21) and references therein. This approach allows one to work with \mathbb{Q} coefficients, which may have some additional applications. Other uses Morse homology as proposed in [RT, PSS]. The necessary analytic details of the latter approach has been established recently in [OZ].

4. SPECTRAL INVARIANTS

The very motivating example of Floer-Novikov complex and its chain level theory was applied by the second named author in the Hamiltonian Floer theory [Oh4, Oh6]. Namely, a spectral number which we denote by $\rho(H; a)$ is associated to $a \in H(M)$ and a Hamiltonian H , and is proved to be independent of various choices, especially of J in [Oh4].

In this section we give a brief summary of this construction. Let $H : S^1 \times M \rightarrow \mathbb{R}$ be a normalized time-dependent nondegenerate Hamiltonian.

Definition 4.1. We put $G(M, \omega) = \{\alpha \cap [\omega] \mid \alpha \in \pi_2(M)\}$.

We define the *action spectrum* of H by

$$\text{Spec}(H) := \{\mathcal{A}_H(\gamma, w) \in \mathbb{R} \mid [\gamma, w] \in \text{Crit}(\mathcal{A}_H)\},$$

i.e., the set of critical values of $\mathcal{A}_H : \mathcal{L}_0(M) \rightarrow \mathbb{R}$.

The set $G(M, \omega)$ is a countable subset of \mathbb{R} which is a subgroup of the additive group of \mathbb{R} as a group. It may or may not be discrete.

Definition 4.2. Let $G \subset \mathbb{R}$ be a submonoid. We say that a subset $G' \subset \mathbb{R}$ is a *G-set* if $g \in G, g' \in G'$ implies $g + g' \in G'$.

With this definition, $\text{Spec}(H)$ is $G(M, \omega)$ -set

Lemma 4.3. *If $\lambda \in \text{Spec}(H)$ and $g \in G(M, \omega)$ then $\lambda \pm g \in \text{Spec}(H)$.*

If H is nondegenerate, then the quotient space $\text{Spec}(H)/G(M, \omega)$ with the above action is a finite set and

$$\#(\text{Spec}(H)/G(M, \omega)) \leq \#\text{Per}(H).$$

Proof. Let $[\gamma, w], [\gamma, w'] \in \text{Crit}(\mathcal{A}_H)$. We glue w and w' along γ and obtain $w \# w'$. Its homology class in $H_2(M; \mathbb{Z})$ is well defined. We have

$$\mathcal{A}_H([\gamma, w]) - \mathcal{A}_H([\gamma, w']) = \int_{w \# w'} \omega.$$

The lemma follows easily from this fact and the fact that $\text{Per}(H)$ is a finite set. \square

The following definition is standard.

Definition 4.4. We say that two one-periodic Hamiltonians H and H' are *homotopic* if $\phi_H^1 = \phi_{H'}^1$ and if there exists $\{H^s\}_{s \in [0,1]}$ a one parameter family of one periodic Hamiltonians such that $H^0 = H$, $H^1 = H'$ and $\phi_H^1 = \phi_{H^s}^1$ for all $s \in [0, 1]$. In this case we denote $H \sim H'$ and denote the set of equivalence classes by $\widetilde{\text{Ham}}(M, \omega)$.

The following lemma was proven in [Sc2, P] in the aspherical case and in [Oh2] for the general case. We provide its proof in Section 10 for reader's convenience.

Proposition 4.5. *Suppose that H, H' are normalized one periodic Hamiltonians. If $H \sim H'$, we have $\text{Spec}(H) = \text{Spec}(H')$ as a subset of \mathbb{R} .*

This enables one to make the following definition

Definition 4.6. We define the spectrum of $\tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$ to be $\text{Spec}(\tilde{\psi}) := \text{Spec}(\underline{H})$ for a (and so any) Hamiltonian H satisfying $\tilde{\psi} = [\phi_H]$.

Here we denote the normalization of H by

$$\underline{H}(t, x) = H(t, x) - \frac{1}{\text{vol}_\omega(M)} \int_M H_t \omega^n.$$

Definition 4.7. Let G be a subset of \mathbb{R} , which is a monoid. We denote by $\Lambda^\downarrow(G)$ the set of all elements

$$\sum a_i q^{\lambda_i} \in \Lambda^\downarrow$$

such that if $a_i \neq 0$ then $-\lambda_i \in G$. We note that $\Lambda^\downarrow(G)$ forms a subring of Λ^\downarrow and $\Lambda^\downarrow(G)$ is a field if G is a subgroup of the (additive) group \mathbb{R} . We write

$$\Lambda^\downarrow(M) = \Lambda^\downarrow(G(M, \omega)).$$

Suppose that H is nondegenerate. We denote by $CF(M, H)$ the set of infinite sums

$$\sum_{[\gamma, w] \in \text{Crit}(\mathcal{A}_H)} a_{[\gamma, w]} [\gamma, w] \tag{4.1}$$

with $a_{[\gamma, w]} \in \mathbb{C}$ such that for any C the set

$$\{[\gamma, w] \mid a_{[\gamma, w]} \neq 0, \mathcal{A}_H(\gamma, w) \geq -C\}$$

is finite. We denote by $F^\lambda CF(M, H)$ the subset of $CF(M, H)$ consisting of elements (4.1) such that $\mathcal{A}_H([\gamma, w]) \leq \lambda$.

Lemma 4.8. (1) $CF(M, H)$ is a vector space over $\Lambda^\downarrow(M)$.
 (2) $\{[\gamma, w_\gamma] \mid \gamma \in \text{Per}(H)\}$ is a basis of $CF(M, H)$ over $\Lambda^\downarrow(M)$.
 (3) We have

$$CF(M, H; \Lambda^\downarrow) \cong CF(M, H) \otimes_{\Lambda^\downarrow(M)} \Lambda^\downarrow.$$

- (4) The Floer boundary operator $\partial_{(H,J)}$ preserves the submodule $CF(M, H) \subset CF(M, H; \Lambda^\downarrow)$.

This is an easy consequence of Lemma 4.3.

Lemma 4.9. *The chain map $\mathcal{P}_{(H_\chi, J_\chi)}$ in (3.18) induces a $\Lambda^\downarrow(M)$ -linear map*

$$\mathcal{P}_{(H_\chi, J_\chi), \#} : C(M; \Lambda^\downarrow(M)) \rightarrow CF(M, H; \Lambda^\downarrow(M))$$

which are chain-homotopic to one another for different choices of χ .

This is immediate from definition of $\mathcal{P}_{(H_\chi, J_\chi), \#}$. Therefore this together with Theorem 3.10 gives rise to an isomorphism

$$\mathcal{P}_{(H_\chi, J_\chi), *}: H_*(M; \Lambda^\downarrow(M)) \cong HF_*(M, H) \quad (4.2)$$

where the right hand side is the homology of $(CF(M, H), \partial)$. This isomorphism does not depend on the choice of χ 's.

The filtration $F^\lambda CF(M, H; \Lambda^\downarrow)$ induces a filtration $F^\lambda CF(M, H)$ on $CF(M, H)$ in an obvious way.

Definition 4.10. (1) Let $\mathfrak{x} \in HF(H, J)$ be any nonzero Floer homology class. We define its *spectral invariant* $\rho(\mathfrak{x})$ by

$$\rho(\mathfrak{x}) = \inf\{\lambda \mid x \in F^\lambda CF(M, H; \Lambda^\downarrow), \partial_{(H,J)}x = 0, [x] = \mathfrak{x}\}.$$

- (2) If $a \in H^*(M; \Lambda^\downarrow(M))$ and H is a nondegenerate time dependent Hamiltonian, we define the *spectral invariant* $\rho(H; a)$ by

$$\rho(H, J; a) = \rho(\mathcal{P}_{(H_\chi, J_\chi), *}(a^\flat)),$$

where the right hand side is defined in (1) and a^\flat is the Poincaré dual of the cohomology class a . See Notations and Conventions (17).

It is proved in [Oh4, Oh6] that $\rho(H, J; a)$ is independent of J . The same can be proved in general under other choices involved in the definition such as the abstract perturbations in the framework of Kuranishi structure. So we omit J from notation and just denote it by $\rho(H; a)$.

We introduce the following standard invariants associated to the Hamiltonian $H : [0, 1] \times M \rightarrow \mathbb{R}$ called the *positive and negative parts of Hofer's norm* $E^\pm(H)$

$$E^+(H) := \int_{t \in S^1} \max_x H_t dt \quad (4.3)$$

$$E^-(H) := \int_{t \in S^1} -\min_x H_t dt \quad (4.4)$$

for any Hamiltonian H . We have the Hofer norm $\|H\| = E^+(H) + E^-(H)$. We like to emphasize that H is not necessarily one-periodic time-dependent family.

Lemma 4.11. *We have*

$$-E^+(H' - H) \leq \rho(H'; a) - \rho(H; a) \leq E^-(H' - H).$$

This lemma enables one to extend, by continuity, the definition of $\rho(H; a)$ to any Hamiltonian $H : S^1 \times M \rightarrow \mathbb{R}$ which is not necessarily nondegenerate. Lemma 4.11 is proved in a generalized form as Theorem 9.1.

The following homotopy invariance is also proved in [Oh4, Oh6, Us1].

Theorem 4.12 (Homotopy invariance). *Suppose H, H' are normalized. If $H \sim H'$ then $\rho(H; a) = \rho(H'; a)$.*

We will prove it in Section 10 for completeness. This homotopy invariance enables one to extend the definition of $\rho(H; a)$ to non-periodic $H : [0, 1] \times M \rightarrow \mathbb{R}$.

Consider the set of smooth functions $\zeta : [0, 1] \rightarrow [0, 1]$ satisfying $\zeta(0) = 0$, $\zeta(1) = 1$ and ζ is constant in a neighborhood of 0 and 1. Note that this set is convex and so contractible. Let $H : [0, 1] \times M \rightarrow \mathbb{R}$ and $\zeta : [0, 1] \rightarrow [0, 1]$ be such a function. Denote

$$H^\zeta(t, x) = \zeta'(t)H(\zeta(t), x).$$

We note that H^ζ may be regarded as a map for $S^1 \times M$ since $H = 0$ in a neighborhood of $\{0, 1\} \times M$. Moreover the above mentioned convexity implies that $H^{\zeta_1} \sim H^{\zeta_2}$. Therefore $\rho(H^{\zeta_1}; a) = \rho(H^{\zeta_2}; a)$ for any such ζ_i . We define the common number to be $\rho(H; a)$. This gives rise to the map

$$\rho : C^\infty([0, 1] \times M, \mathbb{R}) \times (H_*(M; \Lambda^\perp(M)) \setminus \{0\}) \rightarrow \mathbb{R}. \quad (4.5)$$

Its basic properties are summarized in the next theorem. For

$$a = \sum_{g \in G(M, \omega)} q^g a_g, \text{ with } a_g \in H(M; \mathbb{C})$$

we define

$$\mathbf{v}_q(a) := \max\{g \mid a_g \neq 0\}. \quad (4.6)$$

Theorem 4.13. (Oh) *Let (M, ω) be any closed symplectic manifold. Then the map ρ in (4.5) satisfies the following properties: Let $H, H' \in C^\infty([0, 1] \times M, \mathbb{R})$ and $0 \neq a \in H^*(M; \Lambda^\perp(M))$.*

- (1) (Nondegenerate spectrality) $\rho(H; a) \in \text{Spec}(H)$, if $\tilde{\psi}_H$ is nondegenerate.
- (2) (Projective invariance) $\rho(H; \lambda a) = \rho(H; a)$ for any $0 \neq \lambda \in \mathbb{C}$.
- (3) (Normalization shift) For any function $c : [0, 1] \rightarrow \mathbb{R}$, $\rho(H + c(t); a) = \rho(H; a) - \int_0^1 c(t) dt$.
- (4) (Normalization) $\rho(\mathbf{0}; a) = \mathbf{v}_q(a)$ where $\mathbf{0}$ is the identity in $\widetilde{\text{Ham}}(M, \omega)$.
- (5) (Symplectic invariance) $\rho(H \circ \eta; \eta^* a) = \rho(\tilde{\phi}; a)$ for any symplectic diffeomorphism η . In particular, if $\eta \in \text{Symp}_0(M, \omega)$, then we have $\rho(H \circ \eta; a) = \rho(H; a)$.
- (6) (Triangle inequality) $\rho(H \# H'; a \cup_Q b) \leq \rho(H; a) + \rho(H'; b)$, where $a \cup_Q b$ is a quantum cup product.
- (7) (C^0 -hamiltonian continuity) We have

$$-E^+(H' - H) \leq \rho(H'; a) - \rho(H; a) \leq E^-(H' - H).$$

- (8) (Additive triangle inequality) $\rho(H; a + b) \leq \max\{\rho(H; a), \rho(H; b)\}$.

We refer to [OM] for the precise meaning of the C^0 -hamiltonian continuity stated above.

Theorem 4.13 is stated by the second named author [Oh4, Oh6] in the general context but without detailed account on the construction of virtual fundamental classes in the various moduli spaces entering in the proofs. In the present paper, we provide these details in the framework of Kuranishi structures [FO]. A purely algebraic treatment of the statement (1) is given by Usher [Us1].

By considering the normalization $\underline{H}(t, x)$ of $H(t, x)$, we can interpret $\rho(H; a)$ as the invariant of the associated Hamiltonian path ϕ_H by setting

$$\rho(\phi_H; a) := \rho(\underline{H}; a).$$

The invariance of $H \mapsto \rho(H; a)$ under the equivalence relation $H \sim H'$ enables one to push this down to $\widetilde{\text{Ham}}(M, \omega)$ which we denote $\rho(\phi_H; a)$. We denote the resulting map by

$$\rho : \widetilde{\text{Ham}}(M, \omega) \times (H_*(M; \Lambda^\downarrow(M)) \setminus \{0\}) \rightarrow \mathbb{R}. \quad (4.7)$$

Its basic properties are summarized in the next theorem, which are immediate translation of those stated in Theorem 4.13.

Theorem 4.14. *Let (M, ω) be any closed symplectic manifold. Then the map ρ in (4.7) has the following properties: Let $\tilde{\psi}, \tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ and $0 \neq a \in H^*(M; \Lambda^\downarrow(M))$.*

- (1) (Nondegenerate spectrality) $\rho(\tilde{\psi}; a) \in \text{Spec}(\tilde{\psi})$, if $\tilde{\psi}$ is nondegenerate.
- (2) (Projective invariance) $\rho(\tilde{\phi}; \lambda a) = \rho(\tilde{\phi}; a)$ for any $0 \neq \lambda \in \mathbb{C}$.
- (3) (Normalization) We have $\rho(\underline{0}; a) = \mathbf{v}_q(a)$ where $\underline{0}$ is the identity in $\widetilde{\text{Ham}}(M, \omega)$.
- (4) (Symplectic invariance) $\rho(\eta \circ \tilde{\phi} \circ \eta^{-1}; \eta^* a) = \rho(\tilde{\phi}; a)$ for any symplectic diffeomorphism η . In particular, if $\eta \in \text{Symp}_0(M, \omega)$, then we have $\rho(\eta \circ \tilde{\phi} \circ \eta^{-1}; a) = \rho(\tilde{\phi}; a)$.
- (5) (Triangle inequality) $\rho(\tilde{\phi} \circ \tilde{\psi}; a \cup_Q b) \leq \rho(\tilde{\phi}; a) + \rho(\tilde{\psi}; b)$, where $a \cup_Q b$ is a quantum cup product.
- (6) (C^0 -hamiltonian continuity) We have

$$|\rho(\tilde{\phi}; a) - \rho(\tilde{\psi}; a)| \leq \max\{\|\tilde{\phi} \circ \tilde{\psi}^{-1}\|_+, \|\tilde{\phi} \circ \tilde{\psi}^{-1}\|_-\}$$

where $\|\cdot\|_\pm$ is the positive and negative parts of Hofer's pseudo-norm on $\widetilde{\text{Ham}}(M, \omega)$.

- (7) (Additive triangle inequality) $\rho(\tilde{\phi}; a + b) \leq \max\{\rho(\tilde{\phi}; a), \rho(\tilde{\phi}; b)\}$.

Here we explain the meaning of the negative and positive parts of Hofer's norm $\|\tilde{\psi}\|_\pm$. For $\tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$, we define

$$\|\tilde{\psi}\|_\pm = \inf_H \{E^\pm(H) \mid [\phi_H] = \tilde{\psi}\} \quad (4.8)$$

respectively, and the (strong) Hofer norm $\|\tilde{\psi}\|$ is defined by

$$\|\tilde{\psi}\| = \inf_H \{\|H\| \mid [\phi_H^1] = \tilde{\psi}\}. \quad (4.9)$$

There is another norm, sometimes called the *medium Hofer norm*, which is defined by

$$\|\tilde{\psi}\|_{\text{med}} = \|\tilde{\psi}\|_+ + \|\tilde{\psi}\|_-. \quad (4.10)$$

Obviously we have

$$|\rho(\tilde{\psi}; a) - \rho(\tilde{id}; a)| \leq \|\tilde{\psi}\|_{\text{med}} \leq \|\tilde{\psi}\|$$

for all $a \in QH^*(M; \Lambda^\downarrow(M))$. Here \tilde{id} stands for the constant Hamiltonian isotopy at the identity. If $a \in H^*(M; \mathbb{C}) \subset QH^*(M; \Lambda^\downarrow(M))$, we find that

$$|\rho(\tilde{\psi}; a)| \leq \|\tilde{\psi}\|_{\text{med}} \leq \|\tilde{\psi}\|.$$

See the introduction of [Oh5] for the related discussion.

Remark 4.15. There is another important property, that is compatibility with Poincaré duality observed by Entov-Polterovich [EP1] in the case M is semi-positive and M is rational. Those assumptions are removed by Usher [Us3]. We will discuss some enhancement of this point later in Section 15.

We refer readers to the above references for the proof of Theorems 4.14. Later we will prove its enhancement including bulk deformations. Here are some remarks.

Remark 4.16. We like to note that constructions of $\rho(H; a)$ given in [Oh4] can be carried over whether or not H is normalized. We need the normalization only to descend the spectral function $H \mapsto \rho(H; a)$ to the universal covering space $\widetilde{\text{Ham}}(M, \omega)$ as in Theorem 4.14.

Remark 4.17. In [EP1, EP2, EP3], Entov-Polterovich used different sign conventions from the ones [Oh4] and the present paper. If we compare our convention with the one from [EP3], the only difference lies in the definition of Hamiltonian vector field: our definition, which is the same as that of [Oh4], is given by

$$dH = \omega(X_H, \cdot)$$

while [EP3] takes

$$dH = \omega(\cdot, X_H).$$

Therefore by replacing H by $-H$, one has the same set of closed loops as the periodic solutions of the corresponding Hamiltonian vector fields.

This also results in the difference in the definition of action functional: our definition, the same as the one in [Oh4], is given by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int_0^1 H(t, \gamma(t)) dt \quad (4.11)$$

while [EP1] and [EP3] takes

$$- \int w^* \omega + \int_0^1 H(t, \gamma(t)) dt \quad (4.12)$$

as its definition. We denote the definition (4.12) by $\tilde{\mathcal{A}}_H([\gamma, w])$ for the purpose of comparison of the two below.

Therefore *under the change of H by $-H$* , one has the same set of $\text{Crit } \mathcal{A}_H$ and $\text{Crit } \tilde{\mathcal{A}}_H$ with the same action integrals. Since both conventions use the same associated almost Kähler metric $\omega(\cdot, J\cdot)$, the associated perturbed Cauchy-Riemann equations are exactly the same.

In addition, Entov and Polterovich [EP1, EP2] use the notation $c(\mathfrak{a}, H)$ for the spectral numbers where \mathfrak{a} is the quantum *homology* class. Our $\rho(H; a)$ is nothing but

$$\rho(H; a) = c(a^b; \tilde{H}) = c(a^b; \overline{H}) \quad (4.13)$$

where a^b is the homology class Poincaré dual to the cohomology class a and \overline{H} is the inverse Hamiltonian of H given by

$$\overline{H}(t, x) = -H(t, \phi_H^t(x)). \quad (4.14)$$

The second identity of (4.13) follows from the fact that $\tilde{H} \sim \overline{H}$. More precisely, \tilde{H} generates flow

$$\phi_{\tilde{H}} : \phi_H^{1-t} \circ \phi_H^{-1}$$

which can be deformed to $\phi_{\overline{H}} : t \mapsto (\phi_H^t)^{-1}$. In fact the following explicit formula provides such a deformation

$$\phi_s^t = \begin{cases} \phi_H^{s-t} \circ (\phi_H^s)^{-1} & \text{for } 0 \leq t \leq s \\ (\phi_H^t)^{-1} & \text{for } s \leq t \leq 1 \end{cases} \quad (4.15)$$

for $0 \leq s \leq 1$. (See the proof of [Oh5] Lemma 5.2 for this formula.)

With these understood, one can translate every statements in [EP1, EP2] into the ones in terms of our notations.

Part 2. Bulk deformations of Hamiltonian Floer homology and spectral invariants

In this chapter, we deform Hamiltonian Floer homology by the element $\mathfrak{b} \in H^{even}(M, \Lambda_0)$ in a way similar to the case of Lagrangian Floer theory in [FOOO1] Section 3.8. We will denote the resulting \mathfrak{b} -deformation by $HF_*^{\mathfrak{b}}(H, J_0; \Lambda^\downarrow)$. As a Λ^\downarrow -module, it is isomorphic to the singular homology $H_*(M; \Lambda^\downarrow)$ for any \mathfrak{b} . Recall that we regard the de Rham complex as a chain complex (3.19).

Using the filtration we obtain a version of spectral invariants, the spectral invariants with bulk deformation, which contains various new information as we demonstrate later in Chapter 5.

5. BIG QUANTUM COHOMOLOGY RING: REVIEW

In this section, we exclusively denote by J_0 the *time-independent* almost complex structures.

The theory of spectral invariants explained in Chapter 1 is closely related to the (*small*) quantum cohomology. The spectral invariant with bulk we are going to construct is closely related to the *big* quantum cohomology, which we review in this section.

Let (M, ω) be a closed symplectic manifold and J_0 a compatible (time independent) almost complex structure. For $\alpha \in H_2(M; \mathbb{Z})$ let $\mathcal{M}_\ell^{\text{cl}}(\alpha; J_0)$ be the moduli space of stable maps from genus zero semi-stable J_0 -holomorphic curves with ℓ marked points and of homology class α . There exists an evaluation map

$$\text{ev} : \mathcal{M}_\ell^{\text{cl}}(\alpha; J_0) \rightarrow M^\ell.$$

The moduli space $\mathcal{M}_\ell^{\text{cl}}(\alpha; J_0)$ has a virtual fundamental cycle and hence defines a class

$$\text{ev}_*[\mathcal{M}_\ell^{\text{cl}}(\alpha; J_0)] \in H_*(M^\ell; \mathbb{Q}).$$

(See [FO].) Here $*$ = $2n + 2c_1(M)(\alpha) + 2\ell - 6$. Let h_1, \dots, h_ℓ be closed differential forms on M such that

$$\sum \deg h_i = 2n + 2c_1(M)(\alpha) + 2\ell - 6. \quad (5.1)$$

We define Gromov-Witten invariant by

$$GW_\ell(\alpha : h_1, \dots, h_\ell) = \int_{\mathcal{M}_\ell^{\text{cl}}(\alpha; J_0)} \text{ev}^*(h_1 \times \dots \times h_\ell) \in \mathbb{R}. \quad (5.2)$$

More precisely, we take multisection \mathfrak{s} of the Kuranishi structure of $\mathcal{M}_\ell^{\text{cl}}(\alpha; J_0)$ and the integration in (5.2) is taken on the zero set of this multisection. (See [FOOO2] Appendix C.) We can prove that (5.2) is independent of the almost complex structure J_0 . We put $GW_\ell(\alpha : h_1, \dots, h_\ell) = 0$ unless (5.1) is satisfied. We now define

$$GW_\ell(h_1, \dots, h_\ell) = \sum_{\alpha} q^{-\alpha \cap \omega} GW(\alpha : h_1, \dots, h_\ell) \in \Lambda^\downarrow. \quad (5.3)$$

By Stokes' theorem ([FOOO2] Lemma C.9) we can prove that $GW_\ell(h_1, \dots, h_\ell)$ depends only on the de Rham cohomology class of h_i and is independent of the closed forms h_i representing de Rham cohomology class.

By extending the definition (5.3) linearly over to a Λ^\downarrow -module homomorphism, we obtain:

$$GW_\ell : H(M; \Lambda^\downarrow)^{\otimes \ell} \rightarrow \Lambda^\downarrow.$$

Definition 5.1. Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0^\perp)$. For each given pair $\mathbf{c}, \mathbf{d} \in H^*(M; \Lambda^\perp)$, we define a product $\mathbf{c} \cup^{\mathbf{b}} \mathbf{d} \in H(M; \Lambda^\perp)$ by the following formula

$$\langle \mathbf{c} \cup^{\mathbf{b}} \mathbf{d}, \mathbf{e} \rangle_{\text{PD}_M} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} GW_{\ell+3}(\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{b}, \dots, \mathbf{b}). \quad (5.4)$$

Here $\langle \cdot, \cdot \rangle_{\text{PD}_M}$ denotes the Poincaré duality. We call $\cup^{\mathbf{b}}$ the *deformed quantum cup product*.

Remark 5.2. We note that the right hand side of (5.4) is an infinite sum. If $\mathbf{b} \in H^{\text{even}}(M; \Lambda^\perp)$, it converges in q -adic topology so (5.4) makes sense. Otherwise we proceed as follows. For the general element $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0^\perp)$, we split

$$\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_2 + \mathbf{b}_+ \quad (5.5)$$

with $\mathbf{b}_0 \in H^0(M; \Lambda_0^\perp)$, $\mathbf{b}_2 \in H^2(M; \mathbb{C})$, and $\mathbf{b}_+ \in H^2(M; \Lambda_-^\perp) \oplus \bigoplus_{k \geq 2} H^{2k}(M; \Lambda_0^\perp)$ and define

$$\langle \mathbf{c} \cup^{\mathbf{b}} \mathbf{d}, \mathbf{e} \rangle_{\text{PD}_M} = \sum_{\ell=0}^{\infty} \sum_{\alpha} \frac{\exp(\mathbf{b}_2 \cap \alpha)}{\ell!} q^{-\alpha \cap \omega} GW_{\ell+3}(\alpha : \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{b}_+, \dots, \mathbf{b}_+). \quad (5.6)$$

We can prove that (5.6) converges in q -adic topology. (This can be proved in the same way as in [FOOO3] Section 9. See [FOOO6] Lemma 2.29.)

Geometrically considering the element $\mathbf{b} \in H^2(M; \Lambda_0^\perp)$ corresponds to twisting the Hamiltonian Floer theory by a B -field and is the analog to Cho's trick of considering nonunitary line bundles [Fu1], [Cho]. (We remark that this q -adic convergence of Gromov-Witten invariant had been known for a long time.)

It is now well-established that $\cup^{\mathbf{b}}$ is associative and graded commutative and is independent of J_0 . We thus obtain a \mathbb{Z}_2 -graded commutative ring

$$QH_b^*(M; \Lambda^\perp) = (H(M; \Lambda^\perp), \cup^{\mathbf{b}}).$$

As we will see later, for the purpose of construction of spectral invariants and of partial symplectic quasistates and quasimorphisms, it is important to use a smaller Novikov ring than Λ . We discuss this point now.

Definition 5.3. Let G be a discrete submonoid of \mathbb{R} . We say an element $\mathbf{b} \in H(M; \Lambda_0^\perp)$ to be G -gapped if \mathbf{b} can be written as

$$\mathbf{b} = \sum_{g \in G} q^{-g} b_g, \quad b_g \in H(M; \mathbb{C}).$$

For each $\mathbf{b} \in H(M; \Lambda_0^\perp)$ there exists a smallest discrete submonoid G such that \mathbf{b} is G -gapped. We write this monoid as $G_0(\mathbf{b})$. Let $G_0(M, \omega)$ be the monoid generated by the set

$$\{\alpha \cap \omega \mid \mathcal{M}_\ell^{\text{cl}}(\alpha; J_0) \neq \emptyset\}.$$

Then $G_0(M, \omega)$ is discrete by the Gromov compactness. Let $G_0(M, \omega, \mathbf{b})$ be the discrete monoid generated by $G_0(M, \omega) \cup G_0(\mathbf{b})$. We define

$$\Lambda_0^\perp(M, \omega, \mathbf{b}) = \left\{ \sum a_i q^{-\lambda_i} \in \Lambda_0^\perp \mid \lambda_i \in G_0(M, \omega, \mathbf{b}) \right\}. \quad (5.7)$$

The following is easy to check.

Lemma 5.4. *The bilinear map $\cup^{\mathbf{b}}$ induces a ring structure on $H(M; \Lambda_0^\perp(M, \omega, \mathbf{b}))$.*

We have thus obtained the associated quantum cohomology ring

$$QH_b(M; \Lambda_0^\downarrow(M, \omega, \mathfrak{b})) = (H(M; \Lambda_0^\downarrow(M, \omega, \mathfrak{b})), \cup^{\mathfrak{b}}). \quad (5.8)$$

Remark 5.5. (1) Via the identification $q = T^{-1}$, we can use

$$\Lambda_0(M, \omega, \mathfrak{b}) = \left\{ \sum_i a_i T^{\lambda_i} \in \Lambda \mid \lambda_i \in G_0(M, \omega; \mathfrak{b}), \lambda_i \rightarrow \infty \right\}$$

in place of $\Lambda_0^\downarrow(M, \omega, \mathfrak{b})$ in (5.8).

- (2) Entov-Polterovich [EP1, EP2, EP3] uses quantum *homology*, where the degree is shifted by $2n$ from the usual degree. The isomorphism in Theorem 3.10 then preserves the degree when we use Conley-Zehnder index as the degree of Floer homology.

Here we use usual degree of quantum *cohomology* and shift the degree of Floer homology by n from Conley-Zehnder index.

In this convention, the (quantum) cup product is (\mathbb{Z}_2) -degree preserving. In ‘quantum homology’, the product of degree d_1 and d_2 classes has degree $d_1 + d_2 - 2n$. We prefer to choose our convention so that product is degree preserving.

6. HAMILTONIAN FLOER HOMOLOGY WITH BULK DEFORMATIONS

In this section we modify the construction of Section 3 and include bulk deformations.

Let $[\gamma, w], [\gamma', w'] \in \text{Crit } \mathcal{A}_H$. Below we will need to consider the moduli space of marked Floer trajectories $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ for each $\ell = 0, 1, \dots$. The moduli space $\mathcal{M}_0(H, J; [\gamma, w], [\gamma', w'])$ coincides with $\mathcal{M}(H, J; [\gamma, w], [\gamma', w'])$ which is defined in Definition 3.4 and Proposition 3.6.

Definition 6.1. We denote by $\widehat{\mathring{\mathcal{M}}}_\ell(H, J; [\gamma, w], [\gamma', w'])$ the set of all $(u; z_1^+, \dots, z_\ell^+)$, where u is a map $u : \mathbb{R} \times S^1 \rightarrow M$ which satisfies Conditions (1) - (4) of Definition 3.4 and z_i^+ ($i = 1, \dots, \ell$) are mutually distinct points of $\mathbb{R} \times S^1$. It carries an \mathbb{R} -action by translations in τ -direction. We denote its quotient space by $\mathring{\mathcal{M}}_\ell(H, J; [\gamma, w], [\gamma', w'])$. We define the evaluation map

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathring{\mathcal{M}}_\ell(H, J; [\gamma, w], [\gamma', w']) \rightarrow M^\ell$$

by

$$\text{ev}(u; z_1^+, \dots, z_\ell^+) = (u(z_1^+), \dots, u(z_\ell^+)).$$

We use the following notation in the next proposition. Denote the set of shuffles of ℓ elements by

$$\text{Shuff}(\ell) = \{(\mathbb{L}_1, \mathbb{L}_2) \mid \mathbb{L}_1 \cup \mathbb{L}_2 = \{1, \dots, \ell\}, \mathbb{L}_1 \cap \mathbb{L}_2 = \emptyset\}. \quad (6.1)$$

For $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$ let $\#\mathbb{L}_i$ be the order of this subset. Then $\#\mathbb{L}_1 + \#\mathbb{L}_2 = \ell$.

Proposition 6.2. (1) *The moduli space $\mathring{\mathcal{M}}_\ell(H, J; [\gamma, w], [\gamma', w'])$ has a compactification $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ that is Hausdorff.*

- (2) *The space $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ has an orientable Kuranishi structure with corners.*

(3) The boundary of $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ is described by

$$\begin{aligned} \partial \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) = \\ \bigcup \mathcal{M}_{\#\mathbb{L}_1}(H, J; [\gamma, w], [\gamma'', w'']) \times \mathcal{M}_{\#\mathbb{L}_2}(H, J; [\gamma'', w''], [\gamma', w']), \end{aligned} \quad (6.2)$$

where the union is taken over all $[\gamma'', w''] \in \text{Crit}(\mathcal{A}_H)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

(4) Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ be the Conley-Zehnder index. Then the (virtual) dimension satisfies the following equality (6.3).

$$\dim \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) = \mu_H([\gamma', w']) - \mu_H([\gamma, w]) - 1 + 2\ell. \quad (6.3)$$

(5) We can define orientations of $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ so that (3) above is compatible with this orientation.

(6) The evaluation map ev extends to a map $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) \rightarrow M^\ell$, which we denote also by ev . It is compatible with (3). Namely if we denote

$$\mathbb{L}_1 = \{i_1, \dots, i_{\#\mathbb{L}_1}\}, \mathbb{L}_2 = \{j_1, \dots, j_{\#\mathbb{L}_2}\}$$

with $i_1 < \dots < i_{\#\mathbb{L}_1}$, $j_1 < \dots < j_{\#\mathbb{L}_2}$, then ev_k of the first factor (resp. the second factor) of the right hand side of (6.2) coincides with ev_{i_k} (resp. ev_{j_k}) of the left hand side of (6.2).

The proof of Proposition 6.2 is the same as that of Proposition 3.6 and so is omitted. We are ready to define the deformed boundary map $\partial_{(H,J)}^b$. We start with defining the following operator:

Definition 6.3. Let $[\gamma, w], [\gamma', w'] \in \text{Crit} \mathcal{A}_H$ and h_i ($i = 1, \dots, \ell$) be differential forms on M . We define $\mathbf{n}_{(H,J);\ell}([\gamma, w], [\gamma', w'])(h_1, \dots, h_\ell) \in \mathbb{C}$ by

$$\mathbf{n}_{(H,J);\ell}([\gamma, w], [\gamma', w'])(h_1, \dots, h_\ell) = \int_{\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])} \text{ev}_1^* h_1 \wedge \dots \wedge \text{ev}_\ell^* h_\ell. \quad (6.4)$$

By definition (6.4) is zero unless

$$\sum_{i=1}^{\ell} \deg h_i \neq \dim \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']),$$

where the right hand side is as in (6.3).

Remark 6.4. In order to define the integration in (6.4) we need to take a multisection of $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ that is transversal to 0. In our situation the integration (6.4) depends on this perturbation since $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ has codimension one boundary. We take a system of multisections of $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ so that it is compatible with the decomposition (6.2) of the boundary $\partial \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$.

We linearly extend the definition of $\mathbf{n}_{(H,J);\ell}([\gamma, w], [\gamma', w'])$ to a Λ^\perp -multilinear map $(\Omega(M) \widehat{\otimes} \Lambda^\perp)^\ell \rightarrow \Lambda^\perp$, which we denote by the same symbol.

Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0^\perp)$ and split $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_2 + \mathbf{b}_+$ as in (5.5). Take closed forms which represent \mathbf{b}_0 , \mathbf{b}_2 , \mathbf{b}_+ and write them by the same symbols. We then define

$\mathbf{n}_{(H,J);\ell}^{\mathbf{b}}([\gamma, w], [\gamma', w']) \in \Lambda^\downarrow$ for $[\gamma, w], [\gamma', w'] \in \text{Crit}(\mathcal{A}_H)$ by

$$\begin{aligned} & \mathbf{n}_{(H,J);\ell}^{\mathbf{b}}([\gamma, w], [\gamma', w']) \\ &= \sum_{\ell=0}^{\infty} \frac{\exp(w' \cap \mathbf{b}_2 - w \cap \mathbf{b}_2)}{\ell!} \mathbf{n}_{(H,J);\ell}([\gamma, w], [\gamma', w']) \underbrace{(\mathbf{b}_+, \dots, \mathbf{b}_+)}_{\ell}. \end{aligned} \quad (6.5)$$

Lemma 6.5. *The right hand side of (6.5) is finite sums.*

Proof. Suppose $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) \neq \emptyset$ and so $\mathcal{M}(H, J; [\gamma, w], [\gamma', w']) \neq \emptyset$. By the energy identity, we obtain

$$E_{(H,J)}(u) = \mathcal{A}_H([\gamma, w]) - \mathcal{A}_H([\gamma', w']) < \infty$$

for any $u \in \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$. It follows from the Gromov-Floer compactness that the set of $[\gamma', w']$ satisfying $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) \neq \emptyset$ is finite. In particular, the difference $|\mu_H([\gamma', w']) - \mu_H([\gamma, w])|$ is bounded.

The summand corresponding to $[\gamma', w']$ and ℓ in the right hand side of (6.5) vanishes unless $2n\ell \leq \dim \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) = \mu_H([\gamma, w]) - \mu_H([\gamma', w']) + 2\ell$, i.e.,

$$\ell(2n - 2) \leq \mu_H([\gamma, w]) - \mu_H([\gamma', w']).$$

Therefore boundedness of $\mu_H([\gamma, w]) - \mu_H([\gamma', w'])$ also implies boundedness of the number of possible choices of ℓ . This finishes the proof. \square

Definition 6.6. We define the deformed Floer boundary map

$$\partial_{(H,J)}^{\mathbf{b}} : CF(M, H; \Lambda^\downarrow) \rightarrow CF(M, H; \Lambda^\downarrow)$$

by

$$\partial_{(H,J)}^{\mathbf{b}}([\gamma, w]) = \sum_{[\gamma', w']} \mathbf{n}_{(H,J)}^{\mathbf{b}}([\gamma, w], [\gamma', w']) [\gamma', w']. \quad (6.6)$$

We point out that the sum in (6.6) may not be a finite sum.

Lemma 6.7. *The right hand side of (6.6) converges in $CF(M, H; \Lambda^\downarrow)$ and $\partial_{(J,H)}^{\mathbf{b}}$ is continuous in q -adic topology.*

Proof. Let E be any real number and $[\gamma', w'] \in \text{Crit } \mathcal{A}_H$. By Gromov-Floer compactness, the number of $[\gamma, w]$ such that $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ is nonempty and

$$\mathcal{A}_H([\gamma, w]) - \mathcal{A}_H([\gamma', w']) < E$$

is finite. The lemma now follows from the definition of convergence in $CF(M, H; \Lambda^\downarrow)$. \square

Combining Proposition 6.2 (3) with Stokes' theorem, we can check

$$\partial_{(H,J)}^{\mathbf{b}} \circ \partial_{(H,J)}^{\mathbf{b}} = 0.$$

Definition 6.8.

$$HF_*^{\mathbf{b}}(M, H, J; \Lambda^\downarrow) = \frac{\text{Ker } \partial_{(H,J)}^{\mathbf{b}}}{\text{Im } \partial_{(H,J)}^{\mathbf{b}}}.$$

Now we take two parameter family $\{(H_\chi, J_\chi)\}_{\tau \in \mathbb{R}}$ as in (3.12) in the proof of Theorem 3.10.

Theorem 6.9. *There exists a Λ^\downarrow -module isomorphism*

$$\mathcal{P}_{(H_\chi, J_\chi), * }^{\mathfrak{b}} : H^{2n-*}(M; \Lambda^\downarrow) \cong H_*(M; \Lambda^\downarrow) \cong HF_*^{\mathfrak{b}}(M, H, J; \Lambda^\downarrow)$$

for all \mathfrak{b} . We call it the *Piunikhin map with bulk*.

Proof. The proof, which we discuss below, is similar to the proof of Theorem 3.10. We recall that we identify the de Rham complex with a chain complex by

$$\Omega_*(M) \otimes \Lambda^\downarrow \cong \Omega^{2n-*}(M) \otimes \Lambda^\downarrow.$$

In this section we only give the definition of the map $\mathcal{P}_{(H_\chi, J_\chi), * }^{\mathfrak{b}}$. In Section 26 we prove that it is indeed an isomorphism.

Definition 6.10. We denote by $\mathring{\mathcal{M}}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ the set of all $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \mathbb{R} \times S^1 \rightarrow M$ and $z_i^+ : i = 1, \dots, \ell$ such that u satisfies (1)-(4) of Definition 3.13 and $z_i^+ \in \mathbb{R} \times S^1$ are mutually distinct.

The assignment $(u; z_1^+, \dots, z_\ell^+) \mapsto (u(z_1^+), \dots, u(z_\ell^+))$ defines an evaluation map

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) = \mathring{\mathcal{M}}_\ell(H_\chi, J_\chi; *, [\gamma, w]) \rightarrow M^\ell.$$

Proposition 6.11. (1) $\mathring{\mathcal{M}}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ has a compactification, denoted by $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])$, that is Hausdorff.

(2) The space $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ has an orientable Kuranishi structure with corners.

(3) The boundary of $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ is described by

$$\begin{aligned} & \partial \mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w]) \\ &= \bigcup \mathcal{M}_{\#\mathbb{L}_1}(H_\chi, J_\chi; *, [\gamma', w']) \times \mathcal{M}_{\#\mathbb{L}_2}(H, J; [\gamma', w'], [\gamma, w]) \end{aligned} \quad (6.7)$$

where the union is taken over all $[\gamma', w'] \in \text{Crit}(\mathcal{A}_H)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

(4) Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ be the Conley-Zehnder index. Then the (virtual) dimension satisfies the following equality:

$$\dim \mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w]) = \mu_H([\gamma, w]) + n + 2\ell. \quad (6.8)$$

(5) We can define orientations of $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ so that (3) above is compatible with this orientation.

(6) The map ev extends to a strongly continuous smooth map $\mathcal{M}_\ell(H_\chi, J_\chi; [\gamma, w]) \rightarrow M^\ell$, which we denote also by ev . It is compatible with (3) in the same sense as Proposition 6.2 (6).

(7) The map $\text{ev}_{-\infty}$ which sends $(u; z_1^+, \dots, z_\ell^+)$ to $\lim_{\tau \rightarrow -\infty} u(\tau, t)$ extends to a strongly continuous smooth map $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w]) \rightarrow M$, which we denote also by $\text{ev}_{-\infty}$. It is compatible with (3).

Proof. The proof of Proposition 6.11 is mostly the same as that of Proposition 3.6. We only need to see that in (6.7) the boundary component such as

$$\mathcal{M}_{\#\mathbb{L}_1}(0, J_0; *, *, C) \times \mathcal{M}_{\#\mathbb{L}_2}(H_\chi, J_\chi; [\gamma, w - C]) \quad (6.9)$$

does not appear. (Here the first factor of (6.9) is a compactified moduli space of the J_0 -holomorphic maps $\mathbb{R} \times S^1 \rightarrow M$ of homotopy class $C \in \pi_2(M)$.)

In fact, the moduli space $\mathcal{M}_{\#\mathbb{L}_1}(0, J_0; *, *, C)$ has an extra S^1 symmetry by the S^1 action of the domain $\mathbb{R} \times S^1$. (See Remark 3.12.) So after taking a quotient

by this S^1 action, this component is of codimension 2. (See the proof of Lemma 26.9.) \square

Let $[\gamma, w] \in \text{Crit } \mathcal{A}_H$ and h_i ($i = 1, \dots, \ell$), and let h be differential forms on M . We take a system of multisections on $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ such that it is compatible with (3). We use it to define (6.10) below. See Remark 6.4. We define $\mathbf{n}_{(H_\chi, J_\chi)}(h; [\gamma, w])(h_1, \dots, h_\ell) \in \mathbb{C}$ by

$$\begin{aligned} & \mathbf{n}_{(H_\chi, J_\chi)}(h; [\gamma, w])(h_1, \dots, h_\ell) \\ &= \int_{\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])} \text{ev}_{-\infty}^* h \wedge \text{ev}_1^* h_1 \wedge \dots \wedge \text{ev}_\ell^* h_\ell. \end{aligned} \quad (6.10)$$

We note that (6.10) is zero by definition unless

$$\deg h + \sum_{i=1}^{\ell} \deg h_i = \dim \mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w]),$$

where the right hand side is as in (6.10). We extend $\mathbf{n}_{(H_\chi, J_\chi)}(h; [\gamma, w])$ to a Λ^\perp -multilinear map $(\Omega(M) \widehat{\otimes} \Lambda^\perp)^\ell \rightarrow \Lambda^\perp$ and denote it by the same symbol.

Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0^\perp)$. We decompose $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_2 + \mathbf{b}_+$ as in (5.5) and regard \mathbf{b}_0 , \mathbf{b}_2 , \mathbf{b}_+ as de Rham (co)homology classes by representing them by closed differential forms. We define an element $\mathbf{n}_{(H_\chi, J_\chi)}^\mathbf{b}(h; [\gamma, w]) \in \Lambda^\perp$ by

$$\mathbf{n}_{(H_\chi, J_\chi)}^\mathbf{b}(h; [\gamma, w]) := \sum_{\ell=0}^{\infty} \frac{\exp(\int w^* \mathbf{b}_2)}{\ell!} \mathbf{n}_{(H_\chi, J_\chi), \ell}([\gamma, w])(h; \underbrace{\mathbf{b}_+, \dots, \mathbf{b}_+}_{\ell}) \quad (6.11)$$

for each given $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$ and a differential form h on M . We can prove that the sum in (6.11) converges in q -adic topology, in the same way as in Lemma 6.5. We now define

$$\mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b}(h) := \sum_{[\gamma, w]} \mathbf{n}_{(H_\chi, J_\chi)}^\mathbf{b}(h; [\gamma, w]) [\gamma, w]. \quad (6.12)$$

We can prove that the right hand side is an element of $CF(H, J; \Lambda^\perp)$ in the same way as in Lemma 6.7. Thus we have defined

$$\mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b} : \Omega_*(M) \widehat{\otimes} \Lambda^\perp \rightarrow CF_*(M, H; \Lambda^\perp).$$

Then the identity

$$\mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b} \circ \partial = \partial_{(H, J)}^\mathbf{b} \circ \mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b} \quad (6.13)$$

is a consequence of (6.7) and Stokes' theorem. We can prove easily that $\mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b}$ are chain homotopic to one another when χ is varied in \mathcal{K} . We denote by

$$\mathcal{P}_{(H_\chi, J_\chi), *}^\mathbf{b} : H_*(M; \Lambda^\perp) \rightarrow HF_*^\mathbf{b}(M, H, J; \Lambda^\perp) \quad (6.14)$$

the map induced on homology. We will prove in Section 26 that it is an isomorphism. \square

7. SPECTRAL INVARIANTS WITH BULK DEFORMATION

We next modify the argument given in Section 4 and define spectral invariants with bulk. Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0)$. We consider *discrete* submonoids $G_0(M, \omega)$ and $G_0(M, \omega, \mathbf{b})$ of \mathbb{R} in Definition 5.3.

Definition 7.1. We denote by $G(M, \omega)$ and $G(M, \omega, \mathbf{b})$ the subgroup of $(\mathbb{R}, +)$ generated by the monoids $G_0(M, \omega)$ and $G_0(M, \omega, \mathbf{b})$, respectively.

We note that $G(M, \omega)$ and $G(M, \omega, \mathfrak{b})$ are not necessarily discrete. We also remark that $G(M, \omega, \mathfrak{b})$ may not be even finitely generated.

Let H be a time-dependent Hamiltonian on M . We defined $\text{Spec}(H)$ in Definition 4.1.

Definition 7.2. We define

$$\begin{aligned} \text{Spec}(H; \mathfrak{b}) &= \text{Spec}(H) + G(M, \omega, \mathfrak{b}) \\ &= \{\lambda_1 + \lambda_2 \mid \lambda_1 \in \text{Spec}(H), \lambda_2 \in G(M, \omega, \mathfrak{b})\}. \end{aligned}$$

For a monoid $G \subset \mathbb{R}$, the ring $\Lambda(G)$ was defined in Definition 4.7.

Definition 7.3. Suppose H is nondegenerate. We put

$$\Lambda_{\mathfrak{b}}^{\downarrow}(M) = \Lambda^{\downarrow}(G(M, \omega, \mathfrak{b}))$$

and

$$CF(M, H; \mathfrak{b}) = CF(M, H) \otimes_{\Lambda^{\downarrow}(M)} \Lambda_{\mathfrak{b}}^{\downarrow}(M).$$

Lemma 7.4. Suppose H is nondegenerate.

- (1) $CF(M, H; \mathfrak{b})$ is a vector space over the field $\Lambda_{\mathfrak{b}}^{\downarrow}(M)$ with a basis $\{[\gamma, w_{\gamma}] \mid \gamma \in \text{Per}(H)\}$.
- (2) If $\mathfrak{x} \in CF(M, H; \mathfrak{b}) \setminus \{0\}$ then $\mathfrak{v}_q(\mathfrak{x}) \in \text{Spec}(H; \mathfrak{b})$.

Proof. Statement (1) follows from the fact that $\mathcal{A}_H([\gamma, w]) - \mathcal{A}_H([\gamma, w']) \in G(M, \omega, \mathfrak{b})$ for $\gamma \in \text{Per}(H)$, $[\gamma, w], [\gamma, w'] \in \text{Crit}(\mathcal{A}_H)$. Then statement (2) follows from statement (1). \square

It is easy to see that the map $\partial_{(H, J)}^{\mathfrak{b}} : CF(M, H; \Lambda^{\downarrow}) \rightarrow CF(M, H; \Lambda^{\downarrow})$ preserves $CF(M, H; \mathfrak{b})$. Moreover the filtration of $CF(M, H; \Lambda^{\downarrow}(M))$ induces one on $CF(M, H; \Lambda_{\mathfrak{b}}^{\downarrow}(M))$ by

$$F^{\lambda}CF(M, H; \Lambda_{\mathfrak{b}}^{\downarrow}(M)) = F^{\lambda}CF(M, H; \Lambda^{\downarrow}(M)) \cap CF(M, H; \Lambda_{\mathfrak{b}}^{\downarrow}(M)).$$

We denote the homology of $(CF(M, H; \mathfrak{b}), \partial_{(H, J)}^{\mathfrak{b}})$ by $HF^{\mathfrak{b}}(M, H, J; \Lambda_{\mathfrak{b}}^{\downarrow}(M))$. Then Lemma 7.4 implies

$$HF^{\mathfrak{b}}(M, H, J; \Lambda^{\downarrow}) \cong HF^{\mathfrak{b}}(M, H, J; \Lambda_{\mathfrak{b}}^{\downarrow}(M)) \otimes_{\Lambda_{\mathfrak{b}}^{\downarrow}(M)} \Lambda^{\downarrow}. \quad (7.1)$$

Therefore Theorem 6.9 implies:

Lemma 7.5. The map $\mathcal{P}_{(H_{\chi}, J_{\chi}), *}^{\mathfrak{b}}$ in (6.14) induces an isomorphism

$$H(M; \Lambda_{\mathfrak{b}}^{\downarrow}(M)) \cong HF^{\mathfrak{b}}(M, H, J; \Lambda_{\mathfrak{b}}^{\downarrow}(M)).$$

Definition 7.6. (1) Let $\mathfrak{x} \in HF^{\mathfrak{b}}(M, H, J; \Lambda^{\downarrow})$. We define its *spectral invariant* $\rho^{\mathfrak{b}}(\mathfrak{x})$ by

$$\rho^{\mathfrak{b}}(\mathfrak{x}) = \inf\{\lambda \mid x \in F^{\lambda}CF(M, H, J; \Lambda^{\downarrow}), \partial_{(H, J)}^{\mathfrak{b}}(x) = 0, [x] = \mathfrak{x} \in HF^{\mathfrak{b}}(M, H, J; \Lambda^{\downarrow})\}.$$

- (2) If $a \in H^*(M; \Lambda_{\mathfrak{b}}^{\downarrow}(M))$ and H is a nondegenerate time-dependent Hamiltonian, we define the *spectral invariant with bulk* $\rho^{\mathfrak{b}}(H; a)$ by

$$\rho^{\mathfrak{b}}(H; a) = \rho^{\mathfrak{b}}(\mathcal{P}_{(H_{\chi}, J_{\chi}), *}^{\mathfrak{b}}(a^{\flat})),$$

where a^{\flat} is the homology class dual to a (see (4.13)) and the right hand side is as in (1), and we regard

$$\mathcal{P}_{(H_{\chi}, J_{\chi}), *}^{\mathfrak{b}}(a^{\flat}) \in HF^{\mathfrak{b}}(M, H, J; \Lambda_{\mathfrak{b}}^{\downarrow}(M)) \subset HF(M, H, J; \Lambda^{\downarrow}).$$

By the same procedure exercised for the spectral invariant $\rho(H; a)$, we can prove that $\rho^{\mathfrak{b}}(\mathcal{P}_{(H_\chi, J_\chi), *}(a^{\mathfrak{b}}))$ do not depend on the choices of J and χ or of other choices involved in the construction of virtual fundamental cycles, and hence $\rho^{\mathfrak{b}}(H; a)$ is well-defined.

Theorem 7.7 (Homotopy invariance). (1) *The spectral invariant $\rho^{\mathfrak{b}}(H; a)$ is independent of the almost complex structure and other choices involved in the definition.*

- (2) *The spectral invariant $\rho^{\mathfrak{b}}(H; a)$ depends only on the homology class of \mathfrak{b} and is independent of the choices of differential forms which represent it.*
 (3) *Suppose $\phi_H^1 = \phi_{H'}^1$ and the paths ϕ_H and $\phi_{H'}$ are homotopic relative to the ends. Then*

$$\rho^{\mathfrak{b}}(H; a) = \rho^{\mathfrak{b}}(H'; a).$$

Theorem 7.7 (1) is proved in Section 9. Theorem 7.7 (3) is proved in Section 10. Theorem 7.7 (2) is proved in Section 27.

Theorem 7.7 implies that the function $H \mapsto \rho^{\mathfrak{b}}(\underline{H}; a)$ descends to the universal covering space $\widetilde{\text{Ham}}(M, \omega)$. We denote by $\rho^{\mathfrak{b}}(\psi_H; a) = \rho^{\mathfrak{b}}(\underline{H}; a)$ if $\widetilde{\psi}_H = [\phi_H] \in \widetilde{\text{Ham}}_{\text{nd}}(M, \omega)$ associated to H as before.

We have thus defined a map

$$\rho^{\mathfrak{b}} : \widetilde{\text{Ham}}_{\text{nd}}(M, \omega) \times (H^*(M; \Lambda_{\mathfrak{b}}^{\downarrow}(M)) \setminus \{0\}) \rightarrow \mathbb{R}. \quad (7.2)$$

It still satisfies the conclusions of Theorem 4.14. Namely we have:

Theorem 7.8. *Let (M, ω) be any closed symplectic manifold. Then the map $\rho^{\mathfrak{b}}$ in (7.2) extends to*

$$\rho^{\mathfrak{b}} : \widetilde{\text{Ham}}(M, \omega) \times (H^*(M; \Lambda_{\mathfrak{b}}^{\downarrow}(M)) \setminus \{0\}) \rightarrow \mathbb{R}. \quad (7.3)$$

It has the following properties. Let $\tilde{\psi}, \tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ and $0 \neq a \in H^(M; \Lambda_{\mathfrak{b}}^{\downarrow}(M))$.*

- (1) (Nondegenerate spectrality) *If $\tilde{\psi}$ is non-degenerate, then $\rho^{\mathfrak{b}}(\tilde{\psi}; a) \in \text{Spec}(H; \mathfrak{b})$.*
- (2) (Projective invariance) *$\rho^{\mathfrak{b}}(\tilde{\phi}; \lambda a) = \rho^{\mathfrak{b}}(\tilde{\phi}; a)$ for any $0 \neq \lambda \in \mathbb{C}$.*
- (3) (Normalization) *We have $\rho^{\mathfrak{b}}(\underline{\mathbf{Q}}; a) = \mathfrak{v}_q(a)$ where $\underline{\mathbf{Q}}$ is the identity in $\widetilde{\text{Ham}}(M, \omega)$ and $\mathfrak{v}_q(a)$ is as in (4.6).*
- (4) (Symplectic invariance) *$\rho^{\eta^* \mathfrak{b}}(\eta \circ \tilde{\phi} \circ \eta^{-1}; \eta^* a) = \rho^{\mathfrak{b}}(\tilde{\phi}; a)$ for any symplectic diffeomorphism η . In particular, if $\eta \in \text{Symp}_0(M, \omega)$, then we have $\rho^{\mathfrak{b}}(\eta \circ \tilde{\phi} \circ \eta^{-1}; a) = \rho^{\mathfrak{b}}(\tilde{\phi}; a)$.*
- (5) (Triangle inequality) *$\rho^{\mathfrak{b}}(\tilde{\phi} \circ \tilde{\psi}; a \cup^{\mathfrak{b}} b) \leq \rho^{\mathfrak{b}}(\tilde{\phi}; a) + \rho^{\mathfrak{b}}(\tilde{\psi}; b)$, where $a \cup^{\mathfrak{b}} b$ is the \mathfrak{b} -deformed quantum cup product.*
- (6) (C^0 -Hamiltonian continuity) *We have*

$$|\rho^{\mathfrak{b}}(\tilde{\phi}; a) - \rho^{\mathfrak{b}}(\tilde{\psi}; a)| \leq \max\{\|\tilde{\phi} \circ \tilde{\psi}^{-1}\|_+, \|\tilde{\phi} \circ \tilde{\psi}^{-1}\|_-\}$$

where $\|\cdot\|_{\pm}$ is the positive and negative parts of Hofer's pseudo-norm on $\widetilde{\text{Ham}}(M, \omega)$. In particular, the function $\rho_a : \tilde{\psi} \mapsto \rho^{\mathfrak{b}}(\tilde{\psi}; a)$ is continuous with respect to the quotient topology under the equivalence relation \sim on the space of Hamiltonian paths $\{\tilde{\psi}_H \mid H \in C^\infty(S^1 \times M, \mathbb{R})\}$.

- (7) (Additive triangle inequality) *$\rho^{\mathfrak{b}}(\tilde{\psi}; a + b) \leq \max\{\rho^{\mathfrak{b}}(\tilde{\psi}; a), \rho^{\mathfrak{b}}(\tilde{\psi}; b)\}$.*

The proofs of Theorems 7.7 and 7.8 occupy the rest of this chapter. Most of the proofs are minor changes of the proofs of Theorem 4.14 in [Oh4, Oh6] and of [Us1].

8. PROOF OF THE SPECTRALITY AXIOM

In this section we prove Theorem 7.8 (1). To include the case when (M, ω) is not rational we use some algebraic results exploited by Usher [Us1]. We reprove a similar result in Subsection 8.1 using the universal Novikov ring.

8.1. Usher's spectrality lemma. Let G be a subgroup of \mathbb{R} . (We do *not* assume that G is discrete.) We define

$$\begin{aligned}\Lambda^\downarrow(G) &= \left\{ \sum_{i=1}^{\infty} a_i q^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \in G, \lim_{i \rightarrow \infty} \lambda_i = -\infty \right\}, \\ \Lambda_0^\downarrow(G) &= \left\{ \sum_{i=1}^{\infty} a_i q^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\leq 0}, \lambda_i \in G, \lim_{i \rightarrow \infty} \lambda_i = -\infty \right\}, \\ \Lambda_-^\downarrow(G) &= \left\{ \sum_{i=1}^{\infty} a_i q^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{< 0}, \lambda_i \in G, \lim_{i \rightarrow \infty} \lambda_i = -\infty \right\}.\end{aligned}$$

It follows that $\Lambda^\downarrow(G)$ is a field of fraction of $\Lambda_0^\downarrow(G)$.

Let \overline{C} be a finite dimensional \mathbb{C} vector space. We put

$$C = \overline{C} \otimes \Lambda^\downarrow, \quad C(G) = \overline{C} \otimes \Lambda^\downarrow(G) \subset C.$$

Let e_i ($i = 1, \dots, N$) be a \mathbb{C} -basis of \overline{C} and λ_i^0 for $i = 1, \dots, N$ be real numbers. We define $\mathbf{v}_q : C \rightarrow \mathbb{R}$ by

$$\mathbf{v}_q \left(\sum_{i=1}^N x_i e_i \right) = \sup \{ \mathbf{v}_q(x_i) + \lambda_i^0 \mid i = 1, \dots, N \},$$

i.e., $\mathbf{v}_q(e_i) = \lambda_i^0$ for $i = 1, \dots, N$. It defines a norm with respect to which C and $C(G)$ are complete. Then we define a G -set

$$G' = \bigcup_{i=1}^N \{ \lambda_i + \lambda \mid \lambda \in G \}. \quad (8.1)$$

It follows from the definition of $\mathbf{v}_q(x)$ that if $x \in C(G)$ then $\mathbf{v}_q(x) \in G'$. We put

$$F^\lambda C = \{ x \in C \mid \mathbf{v}_q(x) \leq \lambda \}, \quad F^\lambda C(G) = F^\lambda C \cap C(G).$$

Suppose that \overline{C} is \mathbb{Z}_2 -graded, i.e., $\overline{C} = \overline{C}^0 \oplus \overline{C}^1$ and each of the element of our basis e_i lies in either \overline{C}^0 or \overline{C}^1 . Let a \mathbb{C} -linear map

$$\partial_g : \overline{C}^i \rightarrow \overline{C}^{i-1}$$

be given for each $g \in G$. Assuming that $\{g \mid \partial_g \neq 0\} \cap \mathbb{R}_{> E}$ is a finite set for any $E \in \mathbb{R}$, we put

$$\partial = \sum_{g \in G} q^g \partial_g : C \rightarrow C.$$

It induces a linear map $C(G) \rightarrow C(G)$, which we also denote by ∂ . If ∂ satisfies $\partial \partial = 0$, (C, ∂) and $(C(G), \partial)$ define chain complexes. Denote by $H(C)$, $H(C(G))$ their homologies respectively, and denote by $H(C(G)) \rightarrow H(C)$ the natural homomorphism induced by $\Lambda^\downarrow(G) \hookrightarrow \Lambda^\downarrow$.

Definition 8.1. For $\mathfrak{x} \in H(C)$, we define the level

$$\rho(\mathfrak{x}) = \inf \{ \mathbf{v}_q(x) \mid x \in C(G), \partial x = 0, [x] = \mathfrak{x} \}.$$

Now the following theorem is proved by Usher [Us1]. Here we give its proof for completeness' sake exploiting the algebraic material developed in Subsection 6.3 of [FOOO1].

Proposition 8.2. (Usher) $\rho(\mathfrak{x}) \in G'$ for any $\mathfrak{x} \in \text{Im}(H(C(G)) \rightarrow H(C))$.

Proof. We first need to slightly modify the discussion in [FOOO1] Subsection 6.3 since the energy level of the basis e_i is not zero but is λ_i^0 here.

We say

$$e_i \sim e_j \quad \text{if and only if} \quad \lambda_i^0 - \lambda_j^0 \in G.$$

By re-choosing the basis $\{e_i\}_{1 \leq i \leq N}$ into the form $\{q^{\mu_i} e_i\}_{1 \leq i \leq N}$ with $\mu_i \in G$ if necessary, we may assume, without loss of generality, that $\lambda_i^0 = \lambda_j^0$ if $e_i \sim e_j$. We assume this in the rest of this subsection.

For each $\lambda \in G'$, define

$$I(\lambda) = \{i \mid \lambda - \lambda_i^0 \in G, 1 \leq i \leq N\}.$$

We denote by $\mu(\lambda)$ the difference $\lambda - \lambda_i^0$ for $i \in I(\lambda)$. By the definition of \sim and the hypothesis we put above, the value $\mu(\lambda)$ is independent of i . We take the direct sum

$$\overline{C}(\lambda) = \bigoplus_{i \in I(\lambda)} \mathbb{C} e_i.$$

Let $x \in C(G)$ be a nonzero element and denote $\lambda = \mathfrak{v}_q(x)$. Then there exists a unique $\sigma(x) \in \overline{C}(\lambda)$ such that

$$\mathfrak{v}_q(x - q^{\mu(\lambda)} \sigma(x)) < \mathfrak{v}_q(x).$$

We call $\sigma(x)$ the *symbol* of x .

Definition 8.3 (Compare [FOOO1] Section 6.3.1). Let $V \subset C(G)$ be a $\Lambda^\downarrow(G)$ vector subspace. A basis $\{e'_i \mid i = 1, \dots, N'\}$ of V is said to be a *standard basis* if the symbols $\{\sigma(e'_i) \mid i = 1, \dots, N'\}$ are linearly independent over \mathbb{C} .

If $\{e'_i \mid i = 1, \dots, N'\}$ is a standard basis, then we have

$$\mathfrak{v}_q \left(\sum_i a_i e'_i \right) = \max \{ \mathfrak{v}_q(a_i) + \mathfrak{v}_q(e'_i) \mid i = 1, \dots, N' \}. \quad (8.2)$$

Lemma 8.4. Any $V \subset C(G)$ has standard basis. Moreover if $V_1 \subset V_2 \subset C$ are $\Lambda^\downarrow(G)$ vector subspaces, then any standard basis of V_1 can be extended to one of V_2 .

Proof. The proof is similar to the proof of [FOOO1] Lemma 6.3.2 and Lemma 6.3.2bis. We give the detail below since we considered Λ in place of $\Lambda^\downarrow(G)$ in [FOOO1].

Let x_1, \dots, x_k be a standard basis of V_1 . We prove the following by induction on ℓ .

Sublemma 8.5. For $\ell \leq \dim V_2 - \dim V_1$, there exists y_1, \dots, y_ℓ such that the set $\{\sigma(x_1), \dots, \sigma(x_k), \sigma(y_1), \dots, \sigma(y_\ell)\}$ is linearly independent over \mathbb{C} .

Proof. The proof is by induction on ℓ . Suppose we have y_1, \dots, y_ℓ as in the sublemma and $\dim V_2 - \dim V_1 \geq \ell + 1$. We will find $y_{\ell+1}$.

Pick $z_1, \dots, z_m \in \overline{C}$ such that $\{\sigma(x_1), \dots, \sigma(x_k), \sigma(y_1), \dots, \sigma(y_\ell), \sigma(z_1), \dots, \sigma(z_m)\}$ is a basis of \overline{C} as a \mathbb{C} -vector space. In particular, $\{x_1, \dots, x_k, y_1, \dots, y_\ell,$

z_1, \dots, z_m is a basis of C as a $\Lambda^\downarrow(G)$ -vector space. We define $A : C(G) \rightarrow C(G)$ a $\Lambda^\downarrow(G)$ -linear isomorphism by

$$A(x_i) = q^{\mu(\mathbf{v}_q(x_i))} \sigma(x_i), \quad A(y_j) = q^{\mu(\mathbf{v}_q(y_j))} \sigma(y_j), \quad A(z_h) = z_h$$

for $i = 1, \dots, k$, $j = 1, \dots, \ell$, $h = 1, \dots, m$. Note that A preserves filtration and $\sigma \circ A = \sigma$. We take $y' \in V_2$ that is linearly independent to $\{x_1, \dots, x_k, y_1, \dots, y_\ell\}$ over $\Lambda^\downarrow(G)$. We write

$$A(y') = \sum_{n=1}^{\infty} q^{\mu(\lambda_n)} \overline{y}'_n$$

where $\overline{y}'_n \in \overline{C}(\lambda_n)$. Note $\mathbf{v}_q(q^{\mu(\lambda_n)} \overline{y}'_n) = \lambda_n$. Moreover, we may assume that $\lambda_n > \lambda_{n+1}$ and $\lim_{n \rightarrow \infty} \lambda_n = -\infty$.

By assumption there exists n such that

$$\overline{y}'_n \notin \bigoplus_{i=1}^k \mathbb{C} \sigma(x_i) \oplus \bigoplus_{j=1}^{\ell} \mathbb{C} \sigma(y_j). \quad (8.3)$$

Let n_0 be the smallest number satisfying (8.3). Put

$$y'' = \sum_{n=n_0}^{\infty} q^{\mu(\lambda_n)} \overline{y}'_n.$$

Clearly, $\sigma(y'')$ is linearly independent to $\sigma(x_1), \dots, \sigma(x_k), \sigma(y_1), \dots, \sigma(y_\ell)$. Hence $y_{\ell+1} = A^{-1}(y'')$ has the required property. \square

Lemma 8.4 follows from Sublemma 8.5 easily. \square

We now consider $\partial : C(G) \rightarrow C(G)$ and its matrix with respect to a basis of $C(G)$. Choose a basis $\{e'_i \mid i = 1, \dots, b\} \cup \{e''_i \mid i = 1, \dots, h\} \cup \{e'''_i \mid i = 1, \dots, b\}$ such that $\{e'_i \mid i = 1, \dots, b\}$ is a standard basis of $\text{Im } \partial$, $\{e'_i \mid i = 1, \dots, b\} \cup \{e''_i \mid i = 1, \dots, h\}$ is a standard basis of $\text{Ker } \partial$ and $\{e'_i \mid i = 1, \dots, b\} \cup \{e''_i \mid i = 1, \dots, h\} \cup \{e'''_i \mid i = 1, \dots, b\}$ is a standard basis of C . (We also assume that e'_i, e''_i, e'''_i are either in C^0 or in C^1 .) Such a basis exists by Lemma 8.4.

Lemma 8.6. *If $a \in H(C(G), \partial)$, there exists a unique $a_i \in \Lambda^\downarrow(G)$ such that $\sum_{i=1}^h a_i e''_i$ represents a . Moreover*

$$\inf\{\mathbf{v}_q(x) \mid x \in \text{Ker } \partial, a = [x]\} = \mathbf{v}_q\left(\sum_{i=1}^h a_i e''_i\right). \quad (8.4)$$

The proof is easy and so omitted.

We note that by the definition (8.1) of G'

$$\mathbf{v}_q\left(\sum_{i=1}^h a_i e''_i\right) \in G'.$$

Proposition 8.2 is proved. \square

Remark 8.7. From the above discussion we have proved

$$\inf\{\mathbf{v}_q(x) \mid x \in C(G), \partial x = 0, [x] = \mathfrak{x}\} = \inf\{\mathbf{v}_q(x) \mid x \in C, \partial x = 0, [x] = \mathfrak{x}\}$$

for $\mathfrak{x} \in \text{Im}(H(C(G)) \rightarrow H(C))$ at the same time.

8.2. Proof of nondegenerate spectrality. In this subsection we apply Proposition 8.2 to prove the following theorem.

Theorem 8.8. *If H is nondegenerate, then $\rho^b(H; a) \in \text{Spec}(H; \mathfrak{b})$.*

Proof. We put $G = G(M, \omega, \mathfrak{b})$. Let \overline{C} be the \mathbb{C} vector space whose basis is given by $\{[\gamma] \mid \gamma \in \text{Per}(H)\}$. Then we have

$$C(G) \cong CF(M, H; \mathfrak{b}), \quad C \cong CF(M, H; \Lambda^\downarrow).$$

In fact, an isomorphism $I : C(G) \cong CF(M, H; \mathfrak{b})$ can be defined by

$$I([\gamma]) = [\gamma, w_\gamma], \tag{8.5}$$

where we take and fix a bounding disc w_γ for each γ .

For each member $e_i = [\gamma_i]$ of the basis of \overline{C} , we put

$$\lambda_i^0 = \mathcal{A}_H([\gamma_i, w_{\gamma_i}]).$$

Then

$$G' = \text{Spec}(H; \mathfrak{b})$$

and the map I preserves filtration. Theorem 8.8 now follows from Proposition 8.2. \square

9. PROOF OF C^0 -HAMILTONIAN CONTINUITY

In this section we prove the following:

Theorem 9.1. *Let $H, H' : S^1 \times M \rightarrow \mathbb{R}$ be smooth functions such that ψ_H and $\psi_{H'}$ are nondegenerate. Let $a \in H(M; \Lambda)$ and $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0)$. Then we have*

$$-E^+(H' - H) \leq \rho^b(H'; a) - \rho^b(H; a) \leq E^-(H' - H). \tag{9.1}$$

Theorem 9.1 together with Theorem 7.7 implies Theorem 7.8 (6). (See the end of Section 10.) We will also prove the following theorem at the same time in this section.

Theorem 9.2. *The value $\rho^b(H, J; a)$ is independent of the choices of J and the abstract perturbations of the moduli space we use during the construction of the number $\rho^b(H, J; a)$.*

Theorem 9.2 is Theorem 7.7 (1).

Proof. The proofs of Theorems 9.1, 9.2 are mostly the same as one presented in [Oh4, Oh5, Oh6]. Let H, H' be in Theorems 9.1 and $J, J' \in j_\omega$. We interpolate them by the family in $\mathcal{P}(j_\omega) = \text{Map}([0, 1], j_\omega)$

$$(F^s, J^s), \quad 0 \leq s \leq 1$$

where $\{J^s\}_{0 \leq s \leq 1}$ with $J^0 = J, J^1 = J'$ and

$$F^s := H + s(H' - H) : S^1 \times M \rightarrow \mathbb{R}. \tag{9.2}$$

(Note $J^s \neq J_s$ where J_s is as in (3.11).) Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be as in Definition 3.11 and elongate the family to the $(\mathbb{R} \times S^1)$ -family (F^χ, J^χ) by

$$F^\chi(\tau, t, x) = F^{\chi(\tau)}(t, x), \quad J_t^\chi = J_t^{\chi(\tau)}.$$

Using this family, we construct a chain map

$$\mathcal{P}_{(F^\chi, J^\chi), H, H'}^b : (CF(M; H; \Lambda^\downarrow), \partial_{(H, J)}^b) \rightarrow (CF(M; H'; \Lambda^\downarrow), \partial_{(H', J')}^b). \tag{9.3}$$

To simplify the notation, we denote $\mathcal{P}_{(F^\times, J^\times), H, H'}^b$ by $\mathcal{P}_{(F^\times, J^\times)}^b$ when no confusion can occur. Let $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$, $[\gamma', w'] \in \text{Crit}(\mathcal{A}_{H'})$.

Definition 9.3. We denote by $\mathring{\mathcal{M}}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])$ the set of all maps $u : \mathbb{R} \times S^1 \rightarrow M$ which satisfy the following conditions:

- (1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J^\times \left(\frac{\partial u}{\partial t} - X_{F^\times}(u) \right) = 0. \quad (9.4)$$

- (2) The energy

$$E_{(F^\times, J^\times)}(u) = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J^\times}^2 + \left| \frac{\partial u}{\partial t} - X_{F^\times}(u) \right|_{J^\times}^2 \right) dt d\tau$$

is finite.

- (3) The map u satisfies the following asymptotic boundary condition:

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = \gamma(t), \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = \gamma'(t).$$

- (4) The homotopy class of $w \# u$ is $[w']$, where $\#$ is the obvious concatenation.

- (5) z_i^+ are mutually distinct points in $\mathbb{R} \times S^1$.

The assignment $(u; z_1^+, \dots, z_\ell^+) \mapsto (u(z_1^+), \dots, u(z_\ell^+))$ defines an evaluation map

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathring{\mathcal{M}}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w']) \rightarrow M^\ell.$$

Proposition 9.4. (1) *The moduli space $\mathring{\mathcal{M}}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])$ has a compactification $\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])$ that is Hausdorff.*

- (2) *The space $\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])$ has an orientable Kuranishi structure with corners.*

- (3) *The boundary of $\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])$ is described by*

$$\begin{aligned} & \partial \mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w']) \\ &= \bigcup \mathcal{M}_{\# \mathbb{L}_1}(H, J; [\gamma, w], [\gamma''; w'']) \times \mathcal{M}_{\# \mathbb{L}_2}(F^\times, J^\times; [\gamma'', w''], [\gamma', w']) \\ & \cup \bigcup \mathcal{M}_{\# \mathbb{L}_1}(F^\times, J^\times; [\gamma, w], [\gamma'''; w''']) \times \mathcal{M}_{\# \mathbb{L}_2}(H'; J'; [\gamma''', w'''], [\gamma', w']) \end{aligned} \quad (9.5)$$

where the first union is taken over all $(\gamma'', w'') \in \text{Crit}(\mathcal{A}_H)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$ and the second union is taken over all $(\gamma''', w''') \in \text{Crit}(\mathcal{A}_{H'})$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

- (4) *Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$, $\mu_{H'} : \text{Crit}(\mathcal{A}_{H'}) \rightarrow \mathbb{Z}$, be the Conley-Zehnder indices. Then the (virtual) dimension satisfies the following equality:*

$$\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w']) = \mu_{H'}([\gamma', w']) - \mu_H([\gamma, w]) + 2\ell. \quad (9.6)$$

- (5) *We can define orientations of $\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])$ so that (3) above is compatible with this orientation.*

- (6) *ev extends to a weakly submersive map $\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w']) \rightarrow M^\ell$, which we denote also by ev. It is compatible with (3).*

The proof of Proposition 9.4 is the same as that of Proposition 3.6 and so is omitted.

Definition 9.5. Let $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$, $[\gamma', w'] \in \text{Crit}(\mathcal{A}_{H'})$ and let h_i ($i = 1, \dots, \ell$) be differential forms on M . We define $\mathbf{n}_{(F^\times, J^\times); [\gamma, w], [\gamma', w']} (h_1, \dots, h_\ell) \in \mathbb{C}$ by

$$\mathbf{n}_{(F^\times, J^\times); [\gamma, w], [\gamma', w']} (h_1, \dots, h_\ell) = \int_{\mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w'])} \text{ev}_1^* h_1 \wedge \dots \wedge \text{ev}_\ell^* h_\ell. \quad (9.7)$$

By definition (9.7) is zero if

$$\sum_{i=1}^{\ell} \deg h_i \neq \dim \mathcal{M}_\ell(F^\times, J^\times; [\gamma, w], [\gamma', w']),$$

where the right hand side is as in (9.6). We extend (9.7) to

$$\mathbf{n}_{(F^\times, J^\times); [\gamma, w], [\gamma', w']} : B_\ell(\Omega(M) \widehat{\otimes} \Lambda^\downarrow) \rightarrow \Lambda^\downarrow$$

by Λ^\downarrow linearity.

Note that we need to make appropriate choice of compatible system of multisections in order to define integration in (9.7). See Remark 6.4. We sometimes omit this remark from now on.

Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0^\downarrow)$. We split $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_2 + \mathbf{b}_+$ as in (5.5). We take closed forms which represent \mathbf{b}_0 , \mathbf{b}_2 , \mathbf{b}_+ and regard them as differential forms. Define $\mathbf{n}_{(F^\times, J^\times)}^\mathbf{b}([\gamma, w], [\gamma', w']) \in \Lambda^\downarrow$ by the sum

$$\mathbf{n}_{(F^\times, J^\times)}^\mathbf{b}([\gamma, w], [\gamma', w']) = \sum_{\ell=0}^{\infty} \frac{\exp(\int (w')^* \mathbf{b}_2 - \int w^* \mathbf{b}_2)}{\ell!} \mathbf{n}_{(F^\times, J^\times); [\gamma, w], [\gamma', w']}(\underbrace{\mathbf{b}_+, \dots, \mathbf{b}_+}_{\ell}). \quad (9.8)$$

We can prove that the sum in (9.8) converges in q -adic topology, in the same way as in Lemma 6.5. We now define

$$\mathcal{P}_{(F^\times, J^\times)}^\mathbf{b}([\gamma, w]) = \sum_{[\gamma', w']} \mathbf{n}_{(F^\times, J^\times)}^\mathbf{b}([\gamma, w], [\gamma', w']) [\gamma', w']. \quad (9.9)$$

We can also prove that the right hand side is an element of $CF(H', J'; \Lambda^\downarrow)$ in the same way as in Lemma 6.7. Thus we have defined (9.3). Then

$$\mathcal{P}_{(F^\times, J^\times)}^\mathbf{b} \circ \partial_{(H, J)}^\mathbf{b} = \partial_{(H', J')}^\mathbf{b} \circ \mathcal{P}_{(F^\times, J^\times)}^\mathbf{b} \quad (9.10)$$

is a consequence of (9.5) and Stokes' theorem.

Now we would like to study the relationship between the Piunikhin maps $P_{(H_\chi, J_\chi)}^\mathbf{b}$ as we vary (H, J) and the elongation function $\chi \in \mathcal{K}$ given in Definition 3.11. Let $\chi \in \mathcal{K}$ and consider the three maps $\mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b}$, $\mathcal{P}_{(H'_\chi, J'_\chi)}^\mathbf{b}$ and $\mathcal{P}_{(F^\times, J^\times)}^\mathbf{b}$.

Proposition 9.6. $\mathcal{P}_{(F^\times, J^\times)}^\mathbf{b} \circ \mathcal{P}_{(H_\chi, J_\chi)}^\mathbf{b}$ is chain homotopic to $\mathcal{P}_{(H'_\chi, J'_\chi)}^\mathbf{b}$.

Proof. Let J_s, J'_s be as in (3.11) and (F^s, J^s) as in (9.2). For $S \in [1, \infty)$, $\tau \in \mathbb{R}$, we define $G_S(\tau, t, x)$ as follows.

$$G_S(\tau, t, x) = \begin{cases} \chi(\tau + 2S) H_t(x) & \tau \leq 0, \quad S \geq 1 \\ F_t^{\chi(\tau - 2S)}(x) & \tau \geq 0, \quad S \geq 1. \end{cases}$$

We also define $J_S(\tau, t, x)$ by

$$J_S(\tau, t, x) = \begin{cases} J_{\chi(\tau + 2S), t} & \tau \leq 0, \quad S \geq 1 \\ J_t^{\chi(\tau - 2S)} & \tau \geq 0, \quad S \geq 1. \end{cases}$$

We extend G_S to $S \in [0, 1]$ by the following formula.

$$G_S(\tau, t, x) = (1 - S)\chi(\tau)H'(t, x) + SG_1(\tau, t, x).$$

Note that G_S may not be smooth on S at $S = 1$, $\tau \in [-10, 10]$. We modify it on a small neighborhood of this set so that G_S becomes a smooth family. We denote it by the same symbol G_S by an abuse of notation.

We extend J_S to $S \in [0, 1]$ so that the following holds.

- (1) At $S = 0$, $J_S(\tau, t)$ coincides with $J'_{\chi(\tau), t}$.
- (2) J_S is t independent for $\tau < -10$. (It may be S dependent there.)

We denote the family obtained above by

$$(\mathcal{G}, \mathcal{J}) = \{(G_S, J_S)\}_{S \in \mathbb{R}_{\geq 0}}.$$

Now for each $S \in \mathbb{R}_{\geq 0}$, we consider

$$\frac{\partial u}{\partial \tau} + J_S \left(\frac{\partial u}{\partial t} - X_{G_S}(u) \right) = 0 \quad (9.11)$$

and define its moduli space $\mathring{\mathcal{M}}_\ell(G_S, J_S; *, [\gamma, w])$ defined in Definition 6.10. We put

$$\mathring{\mathcal{M}}_\ell(\text{para}; *, [\gamma', w']) = \bigcup_{S \in \mathbb{R}_{\geq 0}} \{S\} \times \mathring{\mathcal{M}}_\ell(G_S, J_S; *, [\gamma', w']). \quad (9.12)$$

- Lemma 9.7.** (1) *The moduli space $\mathring{\mathcal{M}}_\ell(\text{para}; *, [\gamma', w'])$ has a compactification $\mathcal{M}_\ell(\text{para}; *, [\gamma', w'])$ that is Hausdorff.*
- (2) *The space $\mathcal{M}_\ell(\text{para}; *, [\gamma', w'])$ has an orientable Kuranishi structure with corners.*
- (3) *The boundary of $\mathcal{M}_\ell(\text{para}; *, [\gamma', w'])$ is described by the following three types of components.*

$$\bigcup \mathcal{M}_{\#\mathbb{L}_1}(\text{para}; *, [\gamma'', w'']) \times \mathcal{M}_{\#\mathbb{L}_2}(H', J'; [\gamma'', w''], [\gamma', w']) \quad (9.13)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, $[\gamma'', w''] \in \text{Crit}(\mathcal{A}_{H'})$.

$$\bigcup \mathcal{M}_{\#\mathbb{L}_1}(H_\chi, J_\chi; *, [\gamma, w]) \times \mathcal{M}_{\#\mathbb{L}_2}(F^\chi, J^\chi; [\gamma, w], [\gamma', w']) \quad (9.14)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$.

$$\mathcal{M}_\ell(H'_\chi, J'_\chi; *, [\gamma', w']). \quad (9.15)$$

- (4) *Then the (virtual) dimension satisfies the following equality:*

$$\mathcal{M}_\ell(\text{para}; *, [\gamma', w']) = \mu_{H'}([\gamma', w']) + n + 1 + 2\ell. \quad (9.16)$$

- (5) *We can define orientations of $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma', w'])$ so that (3) above is compatible with this orientation.*
- (6) *ev extends to a weakly submersive map $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma', w']) \rightarrow M^\ell$, which we denote also by ev. It is compatible with (3).*
- ev_{-∞} : $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma', w']) \rightarrow L$ can be extended also.*

Proof. The proof is mostly the same as the proof of Proposition 3.6. We only mention how the boundary components are as in (3).

(9.13) appears when there is a bubble to $\tau \rightarrow \infty$. The bubble to $\tau \rightarrow -\infty$ is of codimension 2 by S^1 equivalence. (See the proof of Lemma 26.9.)

(9.14) and (9.15) corresponds to $S \rightarrow \infty$ and $S = 0$ respectively. \square

We use this parameterized moduli space in the same way as in the definition of $\mathcal{P}_{(H_\chi, J_\chi)}^b$ and define a degree one map

$$\mathcal{H}_{(\mathcal{G}, \mathcal{J})}^b : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow CF(M, H'; \Lambda^\downarrow).$$

Lemma 9.7 together with Stokes' theorem and a cobordism argument to derive the equality

$$\partial_{(H', J')}^b \circ \mathcal{H}_{(\mathcal{G}, \mathcal{J})}^b + \mathcal{H}_{(\mathcal{G}, \mathcal{J})}^b \circ \partial = \mathcal{P}_{(F^\chi, J^\chi)}^b \circ \mathcal{P}_{(H_\chi, J_\chi)}^b - P_{(H'_\chi, J'_\chi)}^b. \quad (9.17)$$

Proposition 9.6 follows from (9.17). \square

Next we prove the following bound for the action change.

Lemma 9.8. *If $\mathcal{M}_\ell(F^\chi, J^\chi; [\gamma, w], [\gamma', w'])$ is non-empty,*

$$\mathcal{A}_{H'}([\gamma', w']) - \mathcal{A}_H([\gamma, w]) \leq E^-(H' - H).$$

Proof. Let $u \in \mathring{\mathcal{M}}_\ell(F^\chi, J^\chi; [\gamma, w], [\gamma', w'])$. By the same computation as in the proof of Lemma 3.8, we obtain

$$\begin{aligned} \mathcal{A}_{H'}([\gamma', w']) - \mathcal{A}_H([\gamma, w]) &= -E_{(H, J)}(u) - \int_{\mathbb{R}} \int_{S^1} \chi'(\tau)(H' - H) \circ u(\tau, t) dt d\tau \\ &\leq \int_0^1 -\min_x (H'_t(x) - H_t(x)) dt = E^-(H' - H) \end{aligned}$$

where the inequality follows since $\chi' \geq 0$ and $\int \chi' d\tau = 1$. \square

Now we are in the position to complete the proof of Theorem 9.1. By Lemma 9.8, we have

$$\mathcal{P}_{(F^\chi, J^\chi)}^b (F^\lambda CF(M, H; \Lambda^\downarrow)) \subset F^{\lambda + E^-(H' - H)} CF(M, H'; \Lambda^\downarrow). \quad (9.18)$$

Let $\rho = \rho^b(H; a)$ and $\epsilon > 0$. We take $x \in F^{\rho + \epsilon} CF(M, H; \Lambda^\downarrow)$ which represents $\mathcal{P}_{(H_\chi, J_\chi)}^b(a^b)$. Then the element $\mathcal{P}_{(F^\chi, J^\chi)}^b(x) \in F^{\rho + \epsilon - E^-(H' - H)} CF(M, G; \Lambda^\downarrow)$ represents the Floer homology class $\mathcal{P}_{(F^\chi, J^\chi)}^b \mathcal{P}_{(H_\chi, J_\chi)}^b(a^b) = P_{(H'_\chi, J'_\chi)}^b(a^b)$. (Proposition 9.6). Therefore $\rho^b(H'; a) \leq \rho + \epsilon + E^-(H' - H)$. Since ϵ is an arbitrary positive number, we have

$$\rho^b(H'; a) \leq \rho^b(H; a) + E^-(H' - H).$$

By exchanging the role of H' and H we have

$$\rho^b(H; a) \leq \rho^b(H'; a) + E^+(H' - H).$$

The proof of Theorem 9.1 is complete. \square

We note that Theorem 9.2 follows from the above argument applied to the case $H = H'$ but $J \neq J'$. \square

10. PROOF OF HOMOTOPY INVARIANCE

In this section we prove Theorem 7.7 (3) and Theorem 4.12. Let H^s , $s \in [0, 1]$ be a one parameter family of normalized periodic Hamiltonians $H^s : S^1 \times M \rightarrow \mathbb{R}$ such that:

$$\phi_{H^s}^1 \equiv \psi, \phi_{H^s}^0 \equiv id \quad \text{for all } s \in [0, 1]. \quad (10.1)$$

We assume without loss of generality that $H^s(t, x) \equiv 0$ on a neighborhood of $\{[0]\} \times M \subset S^1 \times M$.

We first define an isomorphism

$$I_s : \text{Crit}(\mathcal{A}_{H^0}) \rightarrow \text{Crit}(\mathcal{A}_{H^s}). \quad (10.2)$$

Let $\gamma \in \text{Per}(\mathcal{A}_{H^0})$. Put $p = \gamma(0)$ and $\gamma_s = z_p^{H^s}$ defined by

$$z_p^{H^s}(t) = \phi_{H^s}^t(p).$$

By (10.1), $z_p^{H^s}(1) = \gamma(1) = \psi(p)$ for all $s \in [0, 1]$. Moreover, we have $z_p^{H^s} \in \text{Per}(\mathcal{A}_{H^s})$. We note that $z_p^{H^0} = \gamma$.

Next let $[\gamma, w] \in \text{Crit}(\mathcal{A}_{H^0})$ be a lifting of γ . By concatenating w with $\bigcup_{\sigma \leq s} \gamma_\sigma$ to obtain $w_s : D^2 \rightarrow M$ such that $w_s|_{\partial D^2} = \gamma_s$. We now define

$$I_s([\gamma, w]) = [\gamma_s, w_s]. \quad (10.3)$$

The following is proved in [Sc2], Proposition 3.1 for the symplectically aspherical case and in [Oh2] in general. The following proof is borrowed from [Oh2]

Proposition 10.1. *Suppose that each H^s is normalized and satisfies (10.1). Then we have*

$$\mathcal{A}_{H^s}(I_s([\gamma, w])) = \mathcal{A}_{H^0}([\gamma, w])$$

for all $s \in [0, 1]$.

Proof. To prove the equality, it is enough to prove

$$\frac{d}{ds} \mathcal{A}_{H^s}(I_s([\gamma, w])) = 0 \quad (10.4)$$

for all $s \in [0, 1]$.

Note that $\mathcal{A}_{H^0}(I_0([\gamma, w])) = \mathcal{A}_{H^0}([\gamma, w])$. Denote $H = H(s, t, x) := H^s(t, x)$ and denote by $K = K(s, t, x)$ the normalized Hamiltonian generating the vector field

$$\frac{\partial \phi_{H^s}^t}{\partial s} \circ (\phi_{H^s}^t)^{-1} =: X_K$$

in s -direction. We compute

$$\frac{d}{ds} \mathcal{A}_{H^s}(I_s([\gamma, w])) = (d\mathcal{A}_{H^s}(I_s([\gamma, w]))) \left(\frac{D}{ds} I_s([\gamma, w]) \right) - \int_0^1 \frac{\partial H}{\partial s}(s, t, \gamma_s(t)) dt.$$

Using that $I_s([\gamma, w]) \in \text{Crit} \mathcal{A}_{H^s}$, this reduces to

$$\frac{d}{ds} \mathcal{A}_{H^s}(I_s([\gamma, w])) = - \int_0^1 \frac{\partial H}{\partial s}(s, t, \gamma_s(t)) dt. \quad (10.5)$$

By (10.1), we have

$$X_K(s, 1, x) = 0 = X_K(s, 0, x)$$

which implies $dK_{s,1} \equiv 0$. Therefore $K_{s,1} \equiv c(s)$ where $c : [0, 1] \rightarrow \mathbb{R}$ is a function of s alone. Then by the normalization condition, we obtain

$$K_{s,1} \equiv 0 \equiv K_{s,0}. \quad (10.6)$$

Lemma 10.2.

$$\frac{\partial H}{\partial s}(s, t, \phi_{H^s}^t(p)) = \frac{\partial}{\partial t} (K(s, t, \phi_{H^s}^t(x)(p))) . \quad (10.7)$$

Proof. The following is proved

$$\frac{\partial K}{\partial t} - \frac{\partial H}{\partial s} - \{H, K\} = 0$$

in [Ba1] Proposition I.1.1 for *normalized* family H^s . By rewriting this into

$$\frac{\partial K}{\partial t} + \{K, H\} = \frac{\partial H}{\partial s}$$

and recalling the definition

$$\{K, H\} = \omega(X_K, X_H) = dK(X_H)$$

of the Poisson bracket (in our convention), it is easy to check that this condition is equivalent to (10.7). Here the exterior differential and the Poisson bracket are taken over M for each fixed (s, t) . \square

Therefore we obtain

$$\begin{aligned} \int_0^1 \frac{\partial H}{\partial s}(s, t, \gamma_s(t)) dt &= \int_0^1 \frac{\partial H}{\partial s}(s, t, \phi_{H^s}^t(p)) dt \\ &= \int_0^1 \frac{\partial}{\partial t} (K(s, t, \phi_{H^s}^t(x)(p))) dt \\ &= K(s, 1, \phi_{H^s}^1(p)) - K(s, 0, \phi_{H^s}^0(p)) \\ &= K(s, 1, \psi(p)) - K(s, 0, p) = 0 \end{aligned}$$

where the last equality comes from (10.6). Substituting this into (10.5), we have finished the proof. \square

The following corollary is immediate.

Corollary 10.3. $\text{Spec}(H^0) = \text{Spec}(H^s)$. Moreover $\text{Spec}(H^0; \mathfrak{b}) = \text{Spec}(H^s; \mathfrak{b})$.

The following lemma is proved for arbitrary (M, ω) by the second named author in [Oh3]. (The corresponding theorem in the aspherical case was proved in [Sc2] generalizing a similar theorem in [HZ].)

Lemma 10.4. *The set $\text{Spec}(H)$ has measure zero for any periodic Hamiltonian H .*

This, together with Lemma 10.4 and the fact that the set $G(M, \omega, \mathfrak{b})$ is countable, implies

Corollary 10.5. $\text{Spec}(H; \mathfrak{b})$ has measure zero for any periodic Hamiltonian H and \mathfrak{b} .

Proof of Theorem 7.7 (3). By Theorem 7.7 (1) which is proved in Section 9, the number $\rho^{\mathfrak{b}}(H^s; a)$ is independent of the choices of J and perturbation. By Theorem 9.1 the function $s \mapsto \rho^{\mathfrak{b}}(H^s; a)$ is continuous. Moreover $\rho^{\mathfrak{b}}(H^0; a)$ is contained in a set $\text{Spec}(H^s; \mathfrak{b})$ that is independent of s and has Lebesgue measure 0. (This independence follows from Corollary 10.3.) Therefore $s \mapsto \rho^{\mathfrak{b}}(H^s; a)$ must be a constant function, as required. \square

Theorem 4.12 is a special case of Theorem 7.7 for $b = 0$.

11. PROOF OF THE TRIANGLE INEQUALITY

In this section we prove Theorem 7.8 (5). The proof is divided into several steps.

11.1. Pants products. In this subsection, we define a product structure of Floer cohomology of periodic Hamiltonian system. It is called the *pants product*. Let $J_1 = \{J_{1,t}\}$, $J_2 = \{J_{2,t}\}$ be S^1 -parametrized families of compatible almost complex structures on M . We assume that

$$J_{1,t} = J_{2,t} = J_0, \quad \text{if } t \text{ is in a neighborhood of } [1] \in S^1. \quad (11.1)$$

Here J_0 is a certain compatible almost complex structure on M . We remark that we have already proved J -independence of the spectral invariant. So we may assume the above condition without loss of generality. (Actually we may also choose $J_{1,t} = J_{2,t} = J_0$ without loss of generality. See Remark 3.1 (2).)

We next take time-dependent Hamiltonians H_1, H_2 . After making the associated Hamiltonian isotopy constant near $t = 0, 1$, we may assume

$$H_{1,t} = H_{2,t} = 0, \quad \text{if } t \text{ is in a neighborhood of } [1] \in S^1. \quad (11.2)$$

The pants product is defined by a chain map

$$\begin{aligned} \mathfrak{m}_2^{\text{cl}} : CF(M, H_1, J_1; \Lambda^\downarrow) \otimes CF(M, H_2, J_2; \Lambda^\downarrow) \\ \rightarrow CF(M, H_1 \# H_2, J_1 \# J_2; \Lambda^\downarrow) \end{aligned} \quad (11.3)$$

where

$$(H_1 \# H_2)(t, x) = \begin{cases} 2H_1(2t, x) & t \leq 1/2, \\ 2H_2(2t - 1, x) & t \geq 1/2 \end{cases} \quad (11.4)$$

and

$$(J_1 \# J_2)(t, x) = \begin{cases} J_1(2t, x) & t \leq 1/2, \\ J_2(2t - 1, x) & t \geq 1/2. \end{cases} \quad (11.5)$$

Remark 11.1. Our definition of $H_1 \# H_2$ is different from those used in [Sc2, Oh4]. But the same definition is found in [ASc2].

It is easy to see that

$$\psi_{H_1 \# H_2} = \psi_{H_2} \circ \psi_{H_1}.$$

In the symplectically aspherical case, the detail of the construction (11.3) is written in [Sc1]. Its generalization to arbitrary symplectic manifold is rather immediate with the virtual fundamental chain technique in the framework of Kuranishi structure [FO]. We treat this construction for the general case here together with its generalization including bulk deformations.

Let $\Sigma = S^2 \setminus \{3 \text{ points}\}$. We choose a function $h : \Sigma \rightarrow \mathbb{R}$ with the following properties:

Condition 11.2. (1) It is proper.

(2) It is a Morse function with a unique critical point z_0 such that $h(z_0) = \frac{1}{2}$.

(3) For $s < \frac{1}{2}$, the preimage $h^{-1}(s)$ is a disjoint union of two S^1 's, and for $\tau > \frac{1}{2}$, $h^{-1}(\tau)$ is one S^1 .

We fix a Riemannian metric on Σ such that Σ is isometric to the three copies of $S^1 \times [0, \infty)$ outside a compact set. Let $\psi_{\nabla h}^t$ be the one parameter subgroup associated to the gradient vector field of h . We put

$$\mathfrak{S} = \{z \in \Sigma \mid \lim_{t \rightarrow \infty} \psi_{\nabla h}^t(z) = z_0, \text{ or } \lim_{t \rightarrow -\infty} \psi_{\nabla h}^t(z) = z_0\}$$

i.e., the union of stable and unstable manifolds of z_0 . Take a diffeomorphism

$$\varphi : \mathbb{R} \times ((0, 1/2) \sqcup (1/2, 1)) \rightarrow \Sigma \setminus \mathfrak{S}$$

such that $h(\varphi(\tau, t)) = \tau$ and put a complex structure j_Σ on Σ with respect to which φ is conformal. Such a complex structure can be chosen by first pushing forward the standard one on $\mathbb{R} \times ((0, 1/2) \sqcup (1/2, 1)) \subset \mathbb{C}$ and extending it to whole Σ . This choice of φ and j_Σ also provides the cylindrical ends near each puncture of Σ .

We define a smooth function $H^\varphi : \Sigma \times M \rightarrow \mathbb{R}$ by:

$$H^\varphi(\varphi(\tau, t), x) = (H_1 \# H_2)(t, x) \quad (11.6)$$

on $\Sigma \setminus \mathfrak{S}$ and extending to \mathfrak{S} by 0. This is consistent with the assumption (11.2).

We define a Σ -parametrized family J^φ of almost complex structures by

$$J_{\varphi(\tau, t)}^\varphi = (J_1 \# J_2)_t.$$

Note that the right hand side is J_0 in a neighborhood of \mathfrak{S} . So we can extend it to \mathfrak{S} .

For $\tau < \frac{1}{2}$, we take the identification

$$h^{-1}(\tau) \cong ([0, 1/2]/\sim) \sqcup ([1/2, 1]/\sim),$$

where $0 \sim 1/2$ and $1/2 \sim 1$. Consider the natural diffeomorphisms

$$\begin{aligned} \varphi_1 & : ([0, 1/2]/\sim) \rightarrow ([0, 1]/\sim); t \mapsto 2t \\ \varphi_2 & : ([1/2, 1]/\sim) \rightarrow ([0, 1]/\sim); t \mapsto 2t - 1. \end{aligned}$$

Then we have the identity

$$(H_1 \# H_2) dt = \varphi_i^*(H_i dt), \quad i = 1, 2. \quad (11.7)$$

This can be easily seen from the definition of $H_1 \# H_2$.

Hereafter in this section, we assume that H_1 , H_2 , $H_1 \# H_2$ are all nondegenerate. Let $[\gamma_1, w_1] \in \text{Crit}(\mathcal{A}_{H_1})$, $[\gamma_2, w_2] \in \text{Crit}(\mathcal{A}_{H_2})$ and $[\gamma_3, w_3] \in \text{Crit}(\mathcal{A}_{H_1 \# H_2})$.

Definition 11.3. We denote by $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1; w_1], [\gamma_2; w_2], [\gamma_3; w_3])$ the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \Sigma \rightarrow M$ and $z_i^+ \in \Sigma$ are marked points, which satisfy the following conditions:

- (1) The map $\bar{u} = u \circ \varphi$ satisfies the equation:

$$\frac{\partial \bar{u}}{\partial \tau} + J^\varphi \left(\frac{\partial \bar{u}}{\partial t} - X_{H^\varphi}(\bar{u}) \right) = 0. \quad (11.8)$$

- (2) The energy

$$E_{(H^\varphi, J^\varphi)}(u) = \frac{1}{2} \int \left(\left| \frac{\partial \bar{u}}{\partial \tau} \right|_{J^\varphi}^2 + \left| \frac{\partial \bar{u}}{\partial t} - X_{H^\varphi}(\bar{u}) \right|_{J^\varphi}^2 \right) dt d\tau$$

is finite.

- (3) It satisfies the following three asymptotic boundary conditions.

$$\lim_{\tau \rightarrow +\infty} u(\varphi(\tau, t)) = \gamma(t).$$

$$\lim_{\tau \rightarrow -\infty} u(\varphi(\tau, t)) = \begin{cases} \gamma_1(2t) & t \leq 1/2, \\ \gamma_2(2t - 1) & t \geq 1/2. \end{cases}$$

- (4) The homotopy class of $(w_1 \sqcup w_2) \# u$ is $[w]$ in $\pi_2(\gamma_3)$. Here $(w_1 \sqcup w_2) \# u$ is the obvious concatenation of w_1 , w_2 and u .
- (5) z_1^+, \dots, z_ℓ^+ are mutually distinct.

We denote by

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathring{\mathcal{M}}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma, w]) \rightarrow M^\ell$$

the evaluation map which associates to $(u; z_1^+, \dots, z_\ell^+)$ the point $(u(z_1^+), \dots, u(z_\ell^+))$.

Remark 11.4. One can write the equation (11.8) in a more invariant fashion into the coordinate independent form

$$(du + P_{H^\varphi}(u))^{(0,1)} = 0$$

where P_{H^φ} is a $u^*(TM)$ -valued one form on Σ and the $(0,1)$ -part is taken with respect to $j_\Sigma(y)$ on $T_y\Sigma$ and $J^\varphi(u(y))$ on $T_{u(y)}M$ at each $y \in \Sigma$. In terms of φ , the pull-back $\varphi^*(P_{H^\varphi})$ can be written as

$$\varphi^*(P_{H^\varphi}) = X_{H_i} dt, \quad i = 1, 2, 3$$

on the ends of Σ near the punctures.

Now we have the following proposition that provides basic properties of the moduli space $\mathring{\mathcal{M}}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$.

Proposition 11.5. (1) $\mathring{\mathcal{M}}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ has a compactification $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ that is Hausdorff.

(2) The space $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ has an orientable Kuranishi structure with corners.

(3) The boundary of $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ is described by union of the following three types of direct products.

$$\mathcal{M}_{\# \mathbb{L}_1}(H_1, J_1; [\gamma_1, w_1], [\gamma'_1, w'_1]) \times \mathcal{M}_{\# \mathbb{L}_2}(H^\varphi, J^\varphi; [\gamma'_1, w'_1], [\gamma_2, w_2], [\gamma_3, w_3]) \quad (11.9)$$

where the union is taken over all $[\gamma'_1, w'_1] \in \text{Crit}(H_1)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\mathcal{M}_{\# \mathbb{L}_1}(H_2, J_2; [\gamma_2, w_2], [\gamma'_2, w'_2]) \times \mathcal{M}_{\# \mathbb{L}_2}(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma'_2, w'_2], [\gamma_3, w_3]) \quad (11.10)$$

where the union is taken over all $[\gamma'_2, w'_2] \in \text{Crit}(H_2)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\begin{aligned} & \mathcal{M}_{\# \mathbb{L}_1}(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma'_3, w'_3]) \\ & \times \mathcal{M}_{\# \mathbb{L}_2}(H_1 \# H_2, J_1 \# J_2; [\gamma'_3, w'_3], [\gamma_3, w_3]) \end{aligned} \quad (11.11)$$

where the union is taken over all $[\gamma'_3, w'_3] \in \text{Crit}(H_1 \# H_2)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

(4) Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ be the Conley-Zehnder index. Then the (virtual) dimension satisfies the following equality:

$$\begin{aligned} & \dim \mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3]) \\ & = \mu_{H_1 \# H_2}([\gamma_3, w_3]) - \mu_{H_1}([\gamma_1, w_1]) - \mu_{H_2}([\gamma_2, w_2]) + 2\ell - n. \end{aligned} \quad (11.12)$$

(5) We can define orientations of $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ so that (3) above is compatible with this orientation.

(6) The map ev extends to a strongly continuous smooth map

$$\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3]) \rightarrow M^\ell,$$

which we denote also by ev . It is compatible with (3).

The proof of Proposition 11.5 is the same as that of Proposition 3.6 and so is omitted. Let $\mathfrak{b} \in H^{even}(M; \Lambda_0^\downarrow)$. We split $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_2 + \mathfrak{b}_+$ as in (5.5). We define $\mathfrak{n}_{H^\varphi, J^\varphi}^\mathfrak{b}([\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3]) \in \Lambda_0^\downarrow$ by

$$\begin{aligned} & \mathfrak{n}_{H^\varphi, J^\varphi}^\mathfrak{b}([\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3]) \\ &= \sum_{\ell=0}^{\infty} \frac{\exp(\mathfrak{b}_2 \cap w_3 - \mathfrak{b}_2 \cap w_2 - \mathfrak{b}_2 \cap w_1)}{\ell!} \\ & \int_{\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])} \text{ev}_1^* \mathfrak{b}_+ \wedge \cdots \wedge \text{ev}_\ell^* \mathfrak{b}_+. \end{aligned} \quad (11.13)$$

We define a system of multisections on various $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ which is compatible with the identification of their boundaries in Proposition 11.5 (3) and use it to define the integration in the right hand side.

Definition 11.6. We put

$$\begin{aligned} & \mathfrak{m}_2^{\text{cl}}([\gamma_1, w_1] \otimes [\gamma_2, w_2]) \\ &= \sum_{[\gamma_3, w_3] \in \text{Crit}(\mathcal{A}_{H_1 \# H_2})} \mathfrak{n}_{H^\varphi, J^\varphi}^\mathfrak{b}([\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])[\gamma_3, w_3]. \end{aligned} \quad (11.14)$$

We can prove that the right hand side of (11.14) converges in $CF((H_1 \# H_2, J_2); \Lambda^\downarrow)$ in the same way as the proof of Lemma 6.7. We have thus defined (11.3).

Lemma 11.7.

$$\partial_{(H_1 \# H_2, J_1 \# J_2)}^\mathfrak{b} \circ \mathfrak{m}_2^{\text{cl}} = \mathfrak{m}_2^{\text{cl}} \circ \left(\partial_{(H_1, J_1)}^\mathfrak{b} \widehat{\otimes} 1 + 1 \widehat{\otimes} \partial_{(H_2, J_2)}^\mathfrak{b} \right).$$

Proof. This is a consequence of Proposition 11.5 (3) and Stokes' theorem. In fact, (11.9), (11.10), (11.11) correspond to $\mathfrak{m}_2^{\text{cl}} \circ \partial_{(H_1, J_1)}^\mathfrak{b} \widehat{\otimes} 1$, $\mathfrak{m}_2^{\text{cl}} \circ \partial_{(H_2, J_2)}^\mathfrak{b}$ and $\partial_{(H_1 \# H_2, J_1 \# J_2)}^\mathfrak{b} \circ \mathfrak{m}_2^{\text{cl}}$, respectively. \square

Thus we have

$$\begin{aligned} \mathfrak{m}_2^{\text{cl}} : HF((H_1, J_1); \Lambda^\downarrow) \otimes HF(M, H_2, J_2; \Lambda^\downarrow) \\ \rightarrow HF(M, H_1 \# H_2, J_1 \# J_2; \Lambda^\downarrow). \end{aligned} \quad (11.15)$$

The next proposition shows that it respects the filtration.

Proposition 11.8. For all $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned} & \mathfrak{m}_2^{\text{cl}}(F^{\lambda_1} CF(M, H_1, J_1; \Lambda^\downarrow) \otimes F^{\lambda_2} CF(M, H_2, J_2; \Lambda^\downarrow)) \\ & \subseteq F^{\lambda_1 + \lambda_2} CF(M, H_1 \# H_2, J_1 \# J_2; \Lambda^\downarrow). \end{aligned}$$

Proof.

Lemma 11.9. If $\mathcal{M}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ is nonempty, then

$$\mathcal{A}_{H_1}([\gamma_1, w_1]) + \mathcal{A}_{H_2}([\gamma_2, w_2]) \geq \mathcal{A}_{H_1 \# H_2}([\gamma_3, w_3]).$$

Proof. Let $(u; z_1^+, \dots, z_k^+) \in \mathring{\mathcal{M}}_\ell(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma_3, w_3])$ and $\tau_0 < 0$. We identify

$$h^{-1}(\tau_0) = S_1^1 \sqcup S_2^1.$$

We denote the restriction of u to S_1^1 by $\gamma_1^{\tau_0}$ and the restriction of u to S_2^1 by $\gamma_2^{\tau_0}$.

We concatenate w_1 with $\cup_{\tau \leq \tau_0} \gamma_1^{\tau_0}$ to obtain $w_1^{\tau_0}$ which bounds $\gamma_1^{\tau_0}$. We define $w_2^{\tau_0}$ in the same way.

In the same way as Lemma 3.8, we derive

$$\begin{aligned}\mathcal{A}_{H_1}([\gamma_1, w_1]) &\geq \mathcal{A}_{H_1}([\gamma_1^{\tau_0}, w_1^{\tau_0}]), \\ \mathcal{A}_{H_2}([\gamma_2, w_2]) &\geq \mathcal{A}_{H_2}([\gamma_2^{\tau_0}, w_2^{\tau_0}]).\end{aligned}\tag{11.16}$$

Next let $\tau_0 > 0$. We denote the restriction of u to $h^{-1}(\tau_0)$ by γ^{τ_0} . We concatenate $w_1 \sqcup w_2$ and the restriction of u to $\{z \in \Sigma \mid h(z) \leq \tau_0\}$ to obtain w^{τ_0} . In the same way as Lemma 3.8, we also derive

$$\mathcal{A}_{H_1 \# H_2}([\gamma^{\tau_0}, w^{\tau_0}]) \geq \mathcal{A}_{H_1 \# H_2}([\gamma_3, w_3]).\tag{11.17}$$

It follows easily from definition that

$$\lim_{\tau_0 \rightarrow 0} (\mathcal{A}_{H_1}([\gamma_1^{\tau_0}, w_1^{\tau_0}]) + \mathcal{A}_{H_2}([\gamma_2^{\tau_0}, w_2^{\tau_0}])) = \lim_{\tau_0 \rightarrow 0} \mathcal{A}_{H_1 \# H_2}([\gamma^{\tau_0}, w^{\tau_0}])).\tag{11.18}$$

Lemma 11.9 follows easily from (11.16), (11.17), (11.18). \square

Proposition 11.8 follows immediately from Lemma 11.9. \square

11.2. Multiplicative property of Piunikhin isomorphism. In this subsection, we prove that the Piunikhin isomorphism interpolates the quantum product $\cup_{\mathbf{b}}$ of QH and the \mathbf{b} -deformed pants product of HF .

Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be as in Definition 3.11. For each $S \in \mathbb{R}$, we define

$$H_S^\varphi(z, x) = \chi(h(z) + S)H^\varphi(z, x)$$

where H^φ is as in (11.6). Similarly we define a family $J_S^\varphi(z)$ so that

$$J_S^\varphi(\varphi(\tau, t)) = J^{\varphi(\tau, t)}(\varphi(\tau + S, t)).$$

Due to the condition $J_t \equiv J_0$ near $t = 0$, this definition smoothly extend to whole Σ .

With this preparation, we prove the following:

Theorem 11.10. *For $a_1, a_2 \in H(M; \Lambda^\perp)$, we have*

$$\mathfrak{m}_2^{\text{cl}}(\mathcal{P}_{((H_1)_\chi, (J_1)_\chi), *}(a_1), \mathcal{P}_{((H_2)_\chi, (J_2)_\chi), *}(a_2)) = \mathcal{P}_{((H_1 \# H_2)_\chi, (J_1 \# J_2)_\chi), *}(a_1 \cup^{\mathbf{b}} a_2).$$

Proof. Let $[\gamma_1, w_1] \in \text{Crit}(\mathcal{A}_{H_1})$, $[\gamma_2, w_2] \in \text{Crit}(\mathcal{A}_{H_2})$ and $[\gamma_3, w_3] \in \text{Crit}(\mathcal{A}_{H_1 \# H_2})$.

Definition 11.11. We denote by $\mathring{\mathcal{M}}_\ell(H_S^\varphi, J_S^\varphi; **, [\gamma, w])$ the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \Sigma \rightarrow M$ and $z_i^+ \in \Sigma$, which satisfy the following conditions:

(1) The map $\overline{u} := u \circ \varphi$ satisfies

$$\frac{\partial \overline{u}}{\partial \tau} + J_S^\varphi \left(\frac{\partial \overline{u}}{\partial t} - X_{H_S^\varphi}(\overline{u}) \right) = 0.\tag{11.19}$$

(2) The energy

$$E_{(H_S^\varphi, J_S^\varphi)} = \frac{1}{2} \int \left(\left| \frac{d\overline{u}}{d\tau} \right|_{J_S^\varphi}^2 + \left| \frac{\partial \overline{u}}{\partial t} - X_{H_S^\varphi}(\overline{u}) \right|_{J_S^\varphi}^2 \right) dt d\tau$$

is finite.

(3) It satisfies the following asymptotic boundary condition.

$$\lim_{\tau \rightarrow +\infty} u(\varphi(\tau, t)) = \gamma(t).$$

(4) The homotopy class of u is congruent to $[w]$ modulo \sim .

(5) z_1^+, \dots, z_ℓ^+ are mutually distinct.

We note that (11.19) and the finiteness of energy imply that there exist $p_1, p_2 \in M$ such that

$$\lim_{\tau \rightarrow -\infty} u(\varphi(\tau, t)) = \begin{cases} p_1 & t < 1/2, \\ p_2 & t > 1/2. \end{cases} \quad (11.20)$$

Therefore the homotopy class of u in $\pi_2(\gamma)$ is defined.

We define the evaluation map

$$\text{ev}_{-\infty} = (\text{ev}_{-\infty,1}, \text{ev}_{-\infty,2}) : \mathring{\mathcal{M}}_\ell(H_S^\varphi, J_S^\varphi; **, [\gamma, w]) \rightarrow M^2$$

by $\text{ev}_{-\infty}(u) = (p_1, p_2)$ where p_1, p_2 are as in (11.20), and

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathring{\mathcal{M}}_\ell(H_S^\varphi, J_S^\varphi; **, [\gamma, w]) \rightarrow M^\ell$$

by $\text{ev}(u; z_1^+, \dots, z_\ell^+) = (u(z_1^+), \dots, u(z_\ell^+))$.

We put

$$\mathring{\mathcal{M}}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w]) = \bigcup_{S \in \mathbb{R}} \{S\} \times \mathring{\mathcal{M}}_\ell(H_S^\varphi, J_S^\varphi; **, [\gamma, w]). \quad (11.21)$$

The evaluation maps $\text{ev}_{-\infty}$ and ev are defined on it in an obvious way.

Proposition 11.12. (1) *The moduli space $\mathring{\mathcal{M}}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w])$ has a compactification $\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w])$ that is Hausdorff.*

(2) *The space $\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w])$ has an orientable Kuranishi structure with corners.*

(3) *The boundary of $\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w])$ is described by the union of following three types of direct or fiber products:*

$$\mathcal{M}_{\#\mathbb{L}_1}(\text{para}; H^\varphi, J^\varphi; **, [\gamma', w']) \times \mathcal{M}_{\#\mathbb{L}_2}(H_1 \# H_2, J_1 \# J_2; [\gamma', w'], [\gamma, w]) \quad (11.22)$$

where the union is taken over all $[\gamma', w'] \in \text{Crit}(H_1 \# H_2)$ and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

The second one is

$$\mathcal{M}_{3+\#\mathbb{L}_1}^{\text{cl}}(\alpha; J_0)_{\text{ev}_3 \times \text{ev}_{-\infty}} \mathcal{M}_{\#\mathbb{L}_2}((H_1 \# H_2)_\chi, (J_1 \# J_2)_\chi; *, [\gamma'; w']). \quad (11.23)$$

Here $\mathcal{M}_{3+\#\mathbb{L}_1}^{\text{cl}}(\alpha; J_0)$ is as in Section 5. The union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$ and α, w' such that the obvious concatenation $\alpha \# w'$ is homotopic to w the fiber product is taken over M .

The third type is

$$\begin{aligned} & (\mathcal{M}_{\#\mathbb{L}_1}((H_1)_\chi, (J_1)_\chi; *, [\gamma_1, w_1]) \times \mathcal{M}_{\#\mathbb{L}_2}((H_2)_\chi, (J_2)_\chi; *, [\gamma_2, w_2])) \\ & \times \mathcal{M}_{\#\mathbb{L}_3}(H^\varphi, J^\varphi; [\gamma_1, w_1], [\gamma_2, w_2], [\gamma, w]). \end{aligned} \quad (11.24)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3)$ the triple shuffle of $\{1, \dots, \ell\}$, and $[\gamma_1, w_1] \in \text{Crit}(\mathcal{A}_{H_1})$, $[\gamma_2, w_2] \in \text{Crit}(\mathcal{A}_{H_2})$.

(4) *Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ be the Conley-Zehnder index. Then the (virtual) dimension satisfies the following equality:*

$$\dim \mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w]) = \mu_{H_1 \# H_2}([\gamma, w]) + 2\ell + 1 + n. \quad (11.25)$$

(5) *We can define orientations of $\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w])$ so that (3) above is compatible with this orientation.*

(6) The map ev extends to a strongly continuous smooth map

$$\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w]) \rightarrow M^\ell,$$

which we denote also by ev . It is compatible with (3).

(7) The map $\text{ev}_{-\infty}$ extends also to a strongly continuous smooth map

$$\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w]) \rightarrow M^2,$$

which we denote by $\text{ev}_{-\infty}$. It is compatible with (3).

Proof. The proof is the same as other similar statements appearing in this and several other previous papers, such as [FO, FOOO1]. So it suffices to see how the boundary of our moduli space appears as in (3).

For each fixed S the boundary of $\mathcal{M}_\ell(H_S^\varphi, J_S^\varphi; **, [\gamma, w])$ is described by (11.22), with para being replaced by S . We note that there is a ‘splitting end’ where ‘bubble’ occurs at $\tau \rightarrow \infty$.

The case $S \rightarrow -\infty$ is described by (11.23). We can prove it as follows. We recall that $\lim_{S \rightarrow -\infty} (H_S^\varphi, J_S^\varphi) = (\mathbb{Q}, J_0)$ where J_0 is *time-independent*. We also remark that the moduli space $\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)$ is identified with the moduli space of $(u; z_1^+, \dots, z_\ell^+)$ such that $\bar{u} = u \circ \varphi$ satisfies the equation:

$$\frac{\partial \bar{u}}{\partial \tau} + J_0 \left(\frac{\partial \bar{u}}{\partial t} \right) = 0 \quad (11.26)$$

and $\int u^* \omega < \infty$, $[u] \sim \alpha$. Therefore the ‘bubble’ which slides to $\tau \rightarrow -\infty$ is described by $\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)$.

The other potential ‘splitting end’ where ‘bubble’ occurs at $\tau \rightarrow -\infty$ has codimension two and do not appear here. (This is because of S^1 symmetry on such a bubble.)

Finally the case $S \rightarrow +\infty$ is described by (11.24). \square

To use $\mathcal{M}_\ell(\text{para}; H^\varphi, J^\varphi; **, [\gamma, w])$ to define an appropriate chain homotopy we need to find a perturbation (multisection) on it which is compatible with the description of its boundary given in Proposition 11.12 (3). Since (11.23) involves fiber product we need to find a perturbation so that ev_3 is a submersion on the perturbed moduli space. We need to use a family of multisections for this purpose. The detail of it is given in [FOOO3] Section 12, etc.

We regard $\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)$ as the compactified moduli space of the pair $(u; z_1^+, \dots, z_\ell^+)$ satisfying (11.26) etc. Then we have a family of perturbation $\mathfrak{s} = \{s^{\mathfrak{w}}\}$ parametrized by $\mathfrak{w} \in W$ where W is certain parameter space that is a manifold equipped with a compact support probability measure with smooth kernel. We use it in the way described in [FOOO3] Section 12 to define a smooth correspondence. Here we use the evaluation maps at $1, 2, 4, \dots, \ell + 3$ marked points as an ‘input’ and the evaluation map at the 3rd marked point as an ‘output’. It gives a map

$$\text{Corr}(\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)) : \Omega(M)^{\otimes(2+\ell)} \rightarrow \Omega(M).$$

Namely

$$\begin{aligned} & \text{Corr}(\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0))(h_1, h_2, h_3, \dots, h_{\ell+2}) \\ &= \text{ev}_3!(\text{ev}_1 \times \text{ev}_2 \times \text{ev}_4 \times \dots \times \text{ev}_{\ell+3})^*(h_1, h_2, h_3, \dots, h_{\ell+2}), \end{aligned}$$

where $\text{ev}_3!$ is the integration along fiber of the perturbed moduli space $\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)^{\mathfrak{s}}$ by the map $\text{ev}_3 : \mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)^{\mathfrak{s}} \rightarrow M$. We extend $\text{Corr}(\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0))$ to

$$\text{Corr}(\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)) : (\Omega(M) \widehat{\otimes} \Lambda^{\downarrow})^{\otimes(2+\ell)} \rightarrow \Omega(M) \widehat{\otimes} \Lambda^{\downarrow}$$

by Λ^{\downarrow} -multilinearity.

Let $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0^{\downarrow})$ and split $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_2 + \mathfrak{b}_+$ as in (5.5). We take closed forms which represent \mathfrak{b}_0 , \mathfrak{b}_2 , \mathfrak{b}_+ and regard them as differential forms. Let $a_1, a_2 \in \Omega(M)$. We put

$$\begin{aligned} & \mathfrak{gw}_{2;\alpha}^{\text{cl}}(a_1, a_2) \\ &= \sum_{\ell} \frac{\exp(\mathfrak{b}_2 \cap \alpha)}{\ell!} \text{Corr}(\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0))(a_1, a_2, \mathfrak{b}_+, \dots, \mathfrak{b}_+). \end{aligned}$$

We then define

$$\mathfrak{gw}_2(a_1, a_2) = \sum_{\alpha} q^{-\alpha \cap \omega} \mathfrak{gw}_{2;\alpha}(a_1, a_2). \quad (11.27)$$

We can easily prove that the right hand side of (11.27) converges in $\Omega(M) \widehat{\otimes} \Lambda^{\downarrow}$. Using the fact that Gromov-Witten invariant is well-defined in the homology level (this follows from the fact that $\mathcal{M}_{3+\ell}^{\text{cl}}(\alpha; J_0)$ has a Kuranishi structure *without boundary*), we can easily show that \mathfrak{gw}_2 induces a product map $\cup^{\mathfrak{b}}$ in the cohomology level.

We now go back to the study of the moduli space $\mathcal{M}_{\ell}(\text{para}; H^{\varphi}, J^{\varphi}; **, [\gamma, w])$. We will define a family of multisections on it by an induction over the energy. We note that we have already defined (a family of) multisections of the moduli spaces which appear in the right hand of Proposition 11.12 (3). The fiber product in (11.23) is transversal to our family of multisections since we take the perturbation of the first factor so that ev_3 is a submersion on the perturbed moduli space. Other products appearing in (11.22) and (11.24) are direct products so the perturbation of each of the factors immediately induce one on the product. Thus we have defined a family of multisections on the boundary. It is compatible at the corners by the inductive construction of multisections. Therefore we can extend it to the whole $\mathcal{M}_{\ell}(\text{para}; H^{\varphi}, J^{\varphi}; **, [\gamma, w])$ by the general theory of Kuranishi structure. We use it to define integration on these moduli spaces below.

We now put

$$\begin{aligned} & \mathfrak{n}_{a_1, a_2, \mathfrak{b}, \text{para}; H^{\varphi}, J^{\varphi}; [\gamma, w]} \\ &= \sum_{\ell} \frac{\exp(\mathfrak{b}_2 \cap \alpha)}{\ell!} \int_{\mathcal{M}_{\ell}(\text{para}; H^{\varphi}, J^{\varphi}; [\gamma, w])} \text{ev}_{-\infty}^*(a_1, a_2) \wedge \text{ev}^*(\mathfrak{b}_+ \wedge \dots \wedge \mathfrak{b}_+). \end{aligned}$$

Definition 11.13.

$$\mathfrak{H}_{H^{\varphi}, J^{\varphi}}^{\mathfrak{b}}(a_1, a_2) = \sum_{[\gamma, w]} \mathfrak{n}_{a_1, a_2, \mathfrak{b}, \text{para}; H^{\varphi}, J^{\varphi}; [\gamma, w]}[\gamma, w].$$

We can prove that the right hand side converges in $CF(M, H_1 \# H_2, J_1 \# J_2; \Lambda^{\downarrow})$ in the same way as the proof of convergence of the right hand side of (3.6).

Lemma 11.14. *We have*

$$\begin{aligned} & \partial_{((H_1 \# H_2)_{\chi}, (J_1 \# J_2)_{\chi})}^{\mathfrak{b}} \circ \mathfrak{H}_{H^{\varphi}, J^{\varphi}}^{\mathfrak{b}} + \mathfrak{H}_{H^{\varphi}, J^{\varphi}}^{\mathfrak{b}}(\partial \widehat{\otimes} 1 + 1 \widehat{\otimes} \partial) \\ &= \mathcal{P}_{((H_1 \# H_2)_{\chi}, (J_1 \# J_2)_{\chi})}^{\mathfrak{b}} \circ \mathfrak{gw}_2 - \mathfrak{m}_2^{\text{cl}} \circ \left(\mathcal{P}_{((H_1)_{\chi}, (J_1)_{\chi})}^{\mathfrak{b}} \otimes \mathcal{P}_{((H_2)_{\chi}, (J_2)_{\chi})}^{\mathfrak{b}} \right). \end{aligned}$$

Proof. Using Proposition 11.12 the lemma follows from Stokes' formula [FOOO3] Lemma 12.13 and composition formula [FOOO3] Lemma 12.15. \square

Theorem 11.10 follows immediately from Lemma 11.14. \square

11.3. Wrap-up of the proof of triangle inequality. Now we prove:

Theorem 11.15. *We assume that $H_1, H_2, H_1 \# H_2$ are nondegenerate. Then for any $a_1, a_2 \in H(M; \Lambda^\downarrow)$ we have:*

$$\rho^b(H_1; a_1) + \rho^b(H_2; a_2) \geq \rho^b(H_1 \# H_2; a_1 \cup^b a_2).$$

Proof. Let $\epsilon > 0$ and $\rho_i = \rho(H_i; a_i; \mathfrak{b})$. Let $x_i \in F^{\rho_i + \epsilon} CF(M, H_i, J_i)$ such that $\partial_{(H_i, J_i)}^b(x_i) = 0$ and $[x_i] = \mathcal{P}_{((H_i)_\chi, (J_i)_\chi), *}^b(a_i) \in HF(M, H_i, J_i)$ ($i = 1, 2$).

By Proposition 11.8 we have

$$\mathfrak{m}_2^{\text{cl}}(x_1, x_2) \in F^{\rho_1 + \rho_2 + 2\epsilon} CF(M, H_1 \# H_2, J_1 \# J_2).$$

By Theorem 11.10 we have

$$[\mathfrak{m}_2^{\text{cl}}(x_1, x_2)] = \mathcal{P}_{((H_1 \# H_2)_\chi, (J_1 \# J_2)_\chi), *}^b(a_1 \cup^b a_2).$$

Therefore by definition

$$\rho(H_1 \# H_2; a_1 \cup^b a_2; \mathfrak{b}) \leq \rho_1 + \rho_2 + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, Theorem 11.15 follows. \square

12. PROOFS OF OTHER AXIOMS

We are now ready to complete the proof of Theorem 7.8.

Note that the proof of Theorem 7.7 (1), (3) has been completed in Section 10 and hence the invariant $\rho^b(\tilde{\phi}; a)$ is well-defined for $\tilde{\phi} \in \widetilde{\text{Ham}}_{\text{nd}}(M; \omega)$.

For general $\tilde{\psi}_H \in \widetilde{\text{Ham}}(M; \omega)$, not necessarily nondegenerate, we take nondegenerate H_i which converges to H in C^0 -sense and take the limit $\lim_{i \rightarrow \infty} \rho^b(\tilde{\psi}_{H_i}; a)$. This limit exists and is independent of H_i by Theorem 9.1. We define this limit to be $\rho^b(\tilde{\phi}; a)$ and have thus defined $\rho^b(\tilde{\phi}; a)$ in general. We prove that it satisfies (1) - (7) of Theorem 7.8.

Statement (1) is Theorem 8.8.

Statement (2) is immediate from definition.

Now let us prove (3). In a way similar to the proof of Lemma 9.8, we prove the following:

Lemma 12.1. *If $\mathcal{M}_\ell(H_\chi, J_\chi; *, [\gamma, w])$ is nonempty, then $\mathcal{A}_H([\gamma, w]) \leq E^-(H)$.*

Therefore if $\mathfrak{v}_q(a) < \lambda$ then

$$\mathcal{P}_{(H_\chi, J_\chi)}^b(a^b) \in F^{\lambda + \|H_i\| - \epsilon} CF(M, H, J).$$

It follows that

$$\rho^b(H; a) \leq \lambda + E^-(H).$$

We apply this inequality to a sequence H_i of Hamiltonians converging to 0 such that $\tilde{\psi}_{H_i}$ are nondegenerate. By taking the limit, we have

$$\rho^b(\mathbb{Q}; a) \leq \lambda.$$

Since this holds for any $\lambda > \mathfrak{v}_q(a)$, $\rho^b(\mathbb{Q}; a) \leq \mathfrak{v}_q(a)$. We refer to Proposition 26.10 in Section 26 for the proof of opposite inequality

$$\rho^b(\mathbb{Q}; a) \geq \mathfrak{v}_q(a). \quad (12.1)$$

Statement (4) is immediate from construction.

Statement (5) is Theorem 11.15 in the nondegenerate case. The general case then follows by an obvious limit argument.

Statement (6) immediately follows from Theorem 9.1.

Statement (7) is obvious from construction. We have thus completed the proof of Theorem 7.8 except the opposite inequality (12.1) which is deferred to Section 26. \square

Part 3. Quasi-states and quasimorphisms via spectral invariants with bulk

In this chapter, we show that Entov-Polterovich's theory can be enhanced by involving spectral invariants with bulk, which we have developed in Chapter 2. The generalization is rather straightforward requiring only a small amount of new ideas. So a large portion of this part is actually a review of the works by Entov-Polterovich and Usher [EP1, EP2, EP3, Os2, Us1, Us3]. (It seems, however, that the proof of Theorem 15.1 below is not written in detail to the level of generality that we provide here.)

13. PARTIAL SYMPLECTIC QUASI-STATES

We start by recalling the definition of Calabi homomorphism. Let $H : [0, 1] \times M$ be a time dependent Hamiltonian and ϕ_H^t the t parameter family of Hamiltonian diffeomorphisms induced by it. We note that we do *not* assume that H is normalized. For an open proper subset $U \subset M$ we define

$$\text{Ham}_U(M, \omega) = \{\psi_H \in \text{Ham}(M, \omega) \mid \text{supp } H_t \subset U \text{ for any } t\}. \quad (13.1)$$

We denote the universal covering space of $\text{Ham}_U(M, \omega)$ by $\widetilde{\text{Ham}}_U(M, \omega)$. Each time dependent Hamiltonian H supported in U determines an element $\psi_H = \phi_H^1 \in \text{Ham}_U(M, \omega)$, together with its lifting $\tilde{\psi}_H = [\phi_H]_U \in \widetilde{\text{Ham}}_U(M, \omega)$. Here $[\cdot]_U$ is the path homotopy class of ϕ_H in $\text{Ham}_U(M, \omega)$. We recall the following lemma due to [Ca], whose proof we omit. (See for example [Ba2] Theorem 4.2.7, [MS] p.328–p.329.)

Lemma 13.1. *If $\text{supp } H_t \subset U$ for all t , then the integral*

$$\int_0^1 dt \int_M H_t \omega^n$$

depends only on $\tilde{\psi}_H \in \widetilde{\text{Ham}}_U(M, \omega)$.

Definition 13.2. We define the homomorphism $\text{Cal}_U : \widetilde{\text{Ham}}_U(M, \omega) \rightarrow \mathbb{R}$ by

$$\text{Cal}_U(\tilde{\psi}_H) = \int_0^1 dt \int_M H_t \omega^n,$$

which is called *Calabi homomorphism*.

This is well-defined by Lemma 13.1.

We next recall the notion of partial symplectic quasi-states introduced by Entov-Polterovich [EP2]. We say that a subset U of M is *displaceable* if there exists $\phi \in \text{Ham}(M, \omega)$ such that $\phi(U) \cap \overline{U} = \emptyset$.

Definition 13.3 ([EP2], [EP3]). A *partial symplectic quasi-states* is defined to be a function $\zeta : C^0(M) \rightarrow \mathbb{R}$ that satisfies the following properties:

- (1) (Lipschitz continuity) $|\zeta(F_1) - \zeta(F_2)| \leq \|F_1 - F_2\|_{C^0}$.
- (2) (Semi-homogeneity) $\zeta(\lambda F) = \lambda \zeta(F)$ for any $F \in C^0(M)$ and $\lambda \in \mathbb{R}_{>0}$.
- (3) (Monotonicity) $\zeta(F_1) \leq \zeta(F_2)$ for $F_1 \leq F_2$.
- (4) (Normalization) $\zeta(1) = 1$.
- (5) (Partial additivity) If two $F_1, F_2 \in C^\infty(M)$ satisfy $\{F_1, F_2\} = 0$ and $\text{supp } F_2$ is displaceable, then $\zeta(F_1 + F_2) = \zeta(F_1)$.
- (6) (Symplectic invariance) $\zeta(F) = \zeta(F \circ \psi)$ for any $\psi \in \text{Symp}_0(M, \omega)$.

- (7) (Vanishing) $\zeta(F) = 0$, provided $\text{supp } F$ is displaceable.
 (8) (Triangle inequality) If $\{F_1, F_2\} = 0$, $\zeta(F_1 + F_2) \geq \zeta(F_1) + \zeta(F_2)$.

The triangle inequality property is required in the definition in [EP3], though it is not in [EP2]. The triangle inequality (8) is different from the one in [EP3] and are adapted to our convention. Namely, for a partial symplectic quasi-state ζ^{EP} in the sense of Entov-Polterovich, $\zeta(H) = -\zeta^{EP}(-H)$ gives a partial symplectic quasi-state in the sense of Definition 13.3. We would like to point out that the above vanishing property (7) is actually an immediate consequence of the axiom, partial additivity (5).

The upshot of Entov-Polterovich's discovery is that the spectral invariant function $H \mapsto \rho(H; 1)$ naturally gives rise to an example of partial symplectic quasi-states, which we denote by ζ_1 . In fact, this *spectral partial quasi-states* is the only known example of such partial symplectic quasi-states so far. We call any such partial symplectic quasi-states constructed out of spectral invariants and its bulk-deformed ones as a whole *spectral partial quasi-states*. The main result of the next section is to generalize Entov-Polterovich's construction of spectral partial (symplectic) quasi-states by involving the spectral invariants with bulk.

Recall that the Lie algebra of $\widetilde{\text{Ham}}(M, \omega)$ or $\text{Ham}(M, \omega)$ can be identified with $C^\infty(M)/\mathbb{R} \cong C^\infty(M)_0$, the set of normalized autonomous Hamiltonian functions. The functional $\zeta_1^\infty = \zeta_1|_{C^\infty(M)}$ is defined on the central extension $C^\infty(M)$ of this Lie algebra.

In fact, ζ_1 can be regarded as a 'linearization' of another nonlinear functional defined on $\widetilde{\text{Ham}}(M, \omega)$ which is the functional $\mu : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$ described in [EP2] section 7. This becomes a genuine quasimorphism under a suitable algebraic condition such as semisimplicity of the quantum cohomology ring of the underlying symplectic manifold (M, ω) . Entov-Polterovich did not name this functional μ . We propose to use the name *Entov-Polterovich pre-quasimorphism*, for the function μ which has the properties established in [EP2] section 7. We recall that the Hofer norm $\|\tilde{\phi}\|$ for $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ is defined by

$$\|\tilde{\phi}\| = \inf \left\{ \|H\| \mid [\phi_H] = \tilde{\phi} \right\}. \quad (13.2)$$

Following [EP2], we define another norm $\|\tilde{\phi}\|_U$, called the *fragmentation norm*.

Definition 13.4. We say $\|\tilde{\phi}\|_U \leq m$ if and only there exists $\tilde{\psi}_i \in \widetilde{\text{Ham}}(M, \omega)$, $\tilde{\phi}_i \in \widetilde{\text{Ham}}_U(M, \omega)$ for $i = 1, \dots, m$ such that

$$\tilde{\phi} = \prod_{i=1}^m (\tilde{\psi}_i \tilde{\phi}_i \tilde{\psi}_i^{-1}).$$

The following fragmentation lemma of Banyaga [Ba1] shows that the norm $\|\tilde{\phi}\|_U$ is always finite.

Lemma 13.5 (Banyaga). *Let $U_i \subset M$ be open sets for $i = 1, \dots, N$, $U = \bigcup_i U_i$, and $\tilde{\phi} \in \widetilde{\text{Ham}}_U(M, \omega)$. Then there exists $\tilde{\phi}_j$ such that $\tilde{\phi}_j \in \widetilde{\text{Ham}}_{U_{i(j)}}(M, \omega)$ for some $i(j) \in \{1, \dots, N\}$ and*

$$\tilde{\phi} = \tilde{\phi}_1 \dots \tilde{\phi}_N.$$

Proof. We give a self contained proof below for the sake of completeness and for readers' convenience. By an obvious induction argument it suffices to consider the

case $N = 2$, namely $U = U_1 \cup U_2$. Let $\phi = \phi_H^1 \in \text{Ham}_U(M, \omega)$. We may assume without loss of generality that $\tilde{\phi} = \tilde{\psi}_H$ and $\|H\|_{C^1} < \epsilon$, where ϵ is a positive number depending only on U_1, U_2 and U to be determined later. (This is because any element of $\widetilde{\text{Ham}}_U(M, \omega)$ is a product of finitely many such $\tilde{\phi}$'s.)

We take a pair of open subsets $U_1'' \subset U_1'$ so that $U_1'' \subset U_1' \subset U_1$, $\overline{U_1''} \subset U_1' \subset \overline{U_1'} \subset U_1$ and $U_1'' \cup U_2 \supset \text{supp } H$.

Let $\eta : M \rightarrow [0, 1]$ be a smooth cut-off function such that $\text{supp } \eta \subset U_1$ and that $\eta = 1$ on U_1' . and put $\phi_1 = \psi_{\eta H}$. It is easy to see that if ϵ is sufficiently small then $\phi_1 = \phi$ on U_1'' , where $\phi \in \text{Ham}_U(M, \omega)$ is the projection of $\tilde{\phi}$. Moreover we may assume that $\phi_1(x) = x$ for $x \notin U_1'' \cup U_2$.

Therefore the support of $\phi_2 = \phi_1^{-1}\phi$ is on U_2 and the support of ϕ_1 is on U_1 . Using the fact that they are C^1 -close to the identity, it follows that $\tilde{\phi}_1\tilde{\phi}_2 = \tilde{\phi}$. \square

Definition 13.6. We call a map $\mu : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$ an *Entov-Polterovich pre-quasimorphism* on $\widetilde{\text{Ham}}(M, \omega)$, if the following conditions are satisfied for $\tilde{\psi}, \tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$.

- (1) (Lipschitz continuity) $|\mu(\tilde{\psi}) - \mu(\tilde{\phi})| \leq C\|\tilde{\psi}\tilde{\phi}^{-1}\|$, where $\|\cdot\|$ is the Hofer norm and C is a constant independent of $\tilde{\psi}, \tilde{\phi}$.
- (2) (Semi-homogeneity) $\mu(\tilde{\phi}^n) = n\mu(\tilde{\phi})$ for any $n \in \mathbb{Z}_{\geq 0}$.
- (3) (Controlled quasi-additivity) If $U \subset M$ is displaceable, then there exists a constant K depending only on U such that

$$|\mu(\tilde{\psi}\tilde{\phi}) - \mu(\tilde{\psi}) - \mu(\tilde{\phi})| < K \min(\|\tilde{\psi}\|_U, \|\tilde{\phi}\|_U).$$

- (4) (Symplectic invariance) $\mu(\tilde{\phi}) = \mu(\psi \circ \tilde{\phi} \circ \psi^{-1})$ for all $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ and $\psi \in \text{Symp}_0(M, \omega)$.
- (5) (Calabi Property) If $U \subset M$ is displaceable, then the restriction of μ to $\widetilde{\text{Ham}}_U(M, \omega)$ coincides with Calabi homomorphism Cal_U .

14. CONSTRUCTION BY SPECTRAL INVARIANT WITH BULK

In this section we describe construction of an example of Entov-Polterovich pre-quasimorphism out of spectral invariants with bulk. Let $\mathfrak{b} \in QH(M; \Lambda_0^\downarrow)$ and $e \neq 0 \in H(M; \Lambda^\downarrow)$ satisfying

$$e \cup^{\mathfrak{b}} e = e. \quad (14.1)$$

An obvious example of such e is $e = 1 \in QH(M; \Lambda_0^\downarrow)$. For given $\tilde{\psi}_H \in \widetilde{\text{Ham}}(M, \omega)$, we consider the limit

$$\mu_e^{\mathfrak{b}}(\tilde{\psi}_H) = \text{vol}_\omega(M) \lim_{n \rightarrow +\infty} \frac{\rho^{\mathfrak{b}}((\tilde{\psi}_H)^n; e)}{n}. \quad (14.2)$$

Recall the relationship

$$\rho^{\mathfrak{b}}(\tilde{\psi}_H; e) := \rho^{\mathfrak{b}}(\underline{H}; e) = \rho^{\mathfrak{b}}(H; e) + \frac{1}{\text{vol}_\omega(M)} \text{Cal}(H)$$

for any Hamiltonian H . In particular, the right hand side does not depend on H as long as $[\tilde{\psi}_H]$ remains the same element of $\widetilde{\text{Ham}}(M, \omega)$.

In particular, if H is a time-independent Hamiltonian and so $\psi_{nH} = (\psi_H)^n$, $\tilde{\psi}_H^n = [\psi_{nH}]$, then (14.2) becomes

$$\mu_e^{\mathfrak{b}}(\tilde{\psi}_H) = \text{vol}_\omega(M) \lim_{n \rightarrow +\infty} \frac{\rho^{\mathfrak{b}}(nH; e)}{n} + \text{Cal}(H). \quad (14.3)$$

We define a (nonlinear) functional $\zeta_e^{\mathfrak{b}} : C^0(M) \rightarrow \mathbb{R}$ by

$$\zeta_e^{\mathfrak{b}}(H) = - \lim_{n \rightarrow \infty} \frac{\rho^{\mathfrak{b}}(nH; e)}{n} \quad (14.4)$$

for $H \in C^\infty(M)$ and then extending to $C^0(M)$ by continuity. Then for any $\tilde{\phi}$ generated by *autonomous* (smooth) Hamiltonian H , whether it is normalized or not, we obtain the relationship

$$\frac{1}{\text{vol}_\omega(M)} \mu_e^{\mathfrak{b}}(\tilde{\psi}_H) = -\zeta_e^{\mathfrak{b}}(H) + \frac{1}{\text{vol}_\omega(M)} \text{Cal}(H). \quad (14.5)$$

If H is normalized, $\text{Cal}(H) = 0$ hence $\mu_e^{\mathfrak{b}}(\tilde{\psi}_H) = -\text{vol}_\omega(M) \zeta_e^{\mathfrak{b}}(H)$.

Theorem 14.1. (1) *The limit (14.2) and (14.4) exist.*

- (2) $\mu_e^{\mathfrak{b}}$ becomes a Entov-Polterovich pre-quasimorphism on $\widetilde{\text{Ham}}(M, \omega)$.
- (3) $\zeta_e^{\mathfrak{b}}$ becomes a partial symplectic quasi-state on M .

Remark 14.2. (1) In case $\mathfrak{b} = 0$, Theorem 14.1 is proved by Entov-Polterovich [EP2].

- (2) Actually in [EP2] several additional assumptions are imposed on (M, ω) . Those assumptions are now removed by Usher [Us1, Us3].
- (3) See also [Us4] for works related to the theme of the present paper.

Proof. We mostly follow the arguments presented in pp.86-88 of [EP2] for the proof. We begin with the following:

Proposition 14.3. *Let $U \subset M$ be an open set and $\phi : M \rightarrow M$ a Hamiltonian diffeomorphism such that $\phi(U) \cap \overline{U} = \emptyset$, and $\tilde{\phi} \in \widetilde{\text{Ham}}(M)$ its lift. Let $\tilde{\psi} \in \widetilde{\text{Ham}}_U(M, \omega)$ and $a \in H(M; \Lambda^\downarrow)$, $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0^\downarrow)$. Then*

$$\rho^{\mathfrak{b}}(\tilde{\phi}\tilde{\psi}; a) = \rho^{\mathfrak{b}}(\tilde{\phi}; a) + \frac{\text{Cal}_U(\tilde{\psi})}{\text{vol}_\omega(M)}. \quad (14.6)$$

Proof. The main idea of the proof of the proposition is due to Ostrover [Os1]. It was used by Entov-Polterovich for the proof of [EP2] Lemma 7.2, which we follow here.

Let $H : [0, 1] \times M \rightarrow \mathbb{R}$ be a Hamiltonian such that $\text{supp } H_t$ is compact and contained in U for any t and that $\tilde{\psi} = [\phi_H]$.

Let $F : [0, 1] \times M \rightarrow \mathbb{R}$ be a normalized Hamiltonian such that $[\phi_F] = \tilde{\phi}$.

By the assumption on ϕ and $\tilde{\psi}$, we find that the fixed point set $\text{Fix}(\phi \circ \phi_H^t)$ is independent of t . We note $\phi_{H^s \# F}^1 = \phi \circ \psi_{H^s}$, where H^s is the Hamiltonian generating the flow $t \mapsto \phi_H^{st}$ defined by

$$H^s(t, x) = sH(st, x)$$

and $\#$ is the concatenation defined as in (11.6). Then by the same way as (10.3), we obtain a one-one correspondence

$$I_s : \text{Crit}(\mathcal{A}_{H^0 \# F}) \rightarrow \text{Crit}(\mathcal{A}_{H^s \# F}).$$

Namely we put $I_s([\gamma, w]) = [\gamma^s, w^s]$, where $\gamma^s(t) = \psi_{H^s \# F}^t(\gamma(0))$, $w^s = w \# v^s$ with $v^s(s, t) = \gamma^{\sigma s}(t)$. For any fixed point p of $\phi = \phi_F^1$, we have $p \notin U$. Hence $\phi_{H^s \# F}(p) \notin U$ for $1 \leq t \leq 1/2$, which implies the following

Lemma 14.4. *For any $[\gamma, w] \in \text{Crit}(\mathcal{A}_{H^0 \# F})$, the number $\mathcal{A}_{H^s \# F}(I_s([\gamma, w]))$ is independent of s .*

We consider the normalization of H^s

$$\underline{H}^s(t, x) = H^s(t, x) - \frac{s}{\text{vol}_\omega(M)} \int_M H_t \omega^n.$$

Then Lemma 14.4 implies

$$\begin{aligned} \mathcal{A}_{\underline{H}^s \# F}(I_s([\gamma, w])) &= \mathcal{A}_{H^s \# F}(I_s([\gamma, w])) + \frac{s}{\text{vol}_\omega(M)} \int \int_M H_t \omega^n dt \\ &= \mathcal{A}_F([\gamma, w]) + \frac{s \text{Cal}_U(H)}{\text{vol}_\omega(M)}. \end{aligned}$$

Therefore

$$\text{Spec}(\tilde{\phi} \circ [\phi_{H^s}]; \mathbf{b}) = \text{Spec}(\tilde{\phi}; \mathbf{b}) + \frac{s \text{Cal}_U(H)}{\text{vol}_\omega(M)}.$$

The function $s \mapsto \rho^{\mathbf{b}}(\tilde{\phi} \circ [\phi_{H^s}]; a) - \frac{s \text{Cal}_U(\tilde{\psi}_H)}{\text{vol}_\omega(M)}$ is continuous and takes values in $\text{Spec}(\tilde{\phi}; \mathbf{b})$, which is a set of measure 0 (see Corollary 10.5.) Therefore it must be constant. This finishes the proof of Proposition 14.3. \square

Let e and \mathbf{b} be as in (14.1).

Definition 14.5. Let A be any displaceable closed subset of M . We define the $\rho_e^{\mathbf{b}}$ -spectral displacement energy $\mathfrak{e}(A; e; \mathbf{b})$ by

$$\mathfrak{e}(A; e; \mathbf{b}) = \inf \{ \rho^{\mathbf{b}}(\tilde{\phi}; e) + \rho^{\mathbf{b}}(\tilde{\phi}^{-1}; e) \mid \tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega), \tilde{\phi}(A) \cap \overline{A} = \emptyset \}. \quad (14.7)$$

Lemma 14.6. *Let $U \subset M$ be an open set which is Hamiltonian displaceable and $\tilde{\psi} \in \widetilde{\text{Ham}}_U(M, \omega)$. Then*

$$\mathfrak{v}_q(e) \leq \rho^{\mathbf{b}}(\tilde{\psi}; e) + \rho^{\mathbf{b}}(\tilde{\psi}^{-1}; e) \leq 2\mathfrak{e}(\overline{U}; e; \mathbf{b}). \quad (14.8)$$

Proof. The following proof is the same as that of [EP2] Lemma 7.2. (14.6) implies

$$\rho(\tilde{\phi} \tilde{\psi}^{-1}; a; \mathbf{b}) = \rho^{\mathbf{b}}(\tilde{\phi}; a) - \frac{\text{Cal}_U(\tilde{\psi})}{\text{vol}_\omega(M)}. \quad (14.9)$$

Theorem 7.8 (3), (5) and (14.1) imply

$$\mathfrak{v}_q(e) = \rho(\mathcal{Q}; e; \mathbf{b}) \leq \rho^{\mathbf{b}}(\tilde{\psi}^{-1}; e) + \rho^{\mathbf{b}}(\tilde{\psi}; e)$$

which proves the first inequality of (14.8).

We also have

$$\begin{aligned} \rho^{\mathbf{b}}(\tilde{\psi}; e) &\leq \rho^{\mathbf{b}}(\tilde{\phi} \tilde{\psi}; e) + \rho^{\mathbf{b}}(\tilde{\phi}^{-1}; e) \\ \rho^{\mathbf{b}}(\tilde{\psi}^{-1}; e) &\leq \rho^{\mathbf{b}}(\tilde{\phi} \tilde{\psi}^{-1}; e) + \rho^{\mathbf{b}}(\tilde{\phi}^{-1}; e). \end{aligned} \quad (14.10)$$

By (14.6), (14.9) and (14.10) we have

$$\begin{aligned} \mathfrak{v}_q(e) &\leq \rho^{\mathfrak{b}}(\tilde{\psi}^{-1}; e) + \rho^{\mathfrak{b}}(\tilde{\psi}; e) \\ &\leq \rho^{\mathfrak{b}}(\tilde{\phi}\tilde{\psi}^{-1}; e) + \rho^{\mathfrak{b}}(\tilde{\phi}\tilde{\psi}; e) + 2\rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; e) \\ &\leq 2\rho^{\mathfrak{b}}(\tilde{\phi}; e) + 2\rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; e). \end{aligned}$$

Since this holds for all $\tilde{\phi}$ displacing U , it follows the second inequality of (14.8). \square

Lemma 14.7. *Suppose U is displaceable and $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$, $\tilde{\psi} \in \widetilde{\text{Ham}}_U(M, \omega)$. Then*

$$\rho^{\mathfrak{b}}(\tilde{\phi}; e) + \rho^{\mathfrak{b}}(\tilde{\psi}; e) - 2\mathfrak{e}(\overline{U}; e; \mathfrak{b}) \leq \rho^{\mathfrak{b}}(\tilde{\phi}\tilde{\psi}; e) \leq \rho^{\mathfrak{b}}(\tilde{\phi}; e) + \rho^{\mathfrak{b}}(\tilde{\psi}; e).$$

Proof. The second inequality follows from Theorem 7.8 (5) and (14.1). The first inequality follows from

$$\begin{aligned} \rho^{\mathfrak{b}}(\tilde{\phi}\tilde{\psi}; e) &\geq \rho^{\mathfrak{b}}(\tilde{\phi}; e) - \rho^{\mathfrak{b}}(\tilde{\psi}^{-1}; e) \\ &\geq \rho^{\mathfrak{b}}(\tilde{\phi}; e) + \rho^{\mathfrak{b}}(\tilde{\psi}; e) - 2\mathfrak{e}(A; e; \mathfrak{b}), \end{aligned}$$

where the first inequality follows from Theorem 7.8 (5) and the second follows from Lemma 14.6. \square

Corollary 14.8. *Let $\tilde{\phi}_1, \dots, \tilde{\phi}_m, \tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$. If $\|\tilde{\phi}_i\|_U = 1$ for $i = 1, \dots, m$, then*

$$|\rho^{\mathfrak{b}}(\tilde{\phi}_1 \cdots \tilde{\phi}_m \tilde{\psi}; e) - \sum_{i=1}^m \rho^{\mathfrak{b}}(\tilde{\phi}_i; e) - \rho^{\mathfrak{b}}(\tilde{\psi}; e)| < 2m\mathfrak{e}(\overline{U}; e; \mathfrak{b}). \quad (14.11)$$

Proof. By the hypothesis $\|\tilde{\phi}_i\|_U = 1$, we can write $\tilde{\phi}_i = \tilde{\phi}_i^{-1} \tilde{\phi}'_i \tilde{\phi}_i$ with $\tilde{\phi}'_i \in \widetilde{\text{Ham}}_U(M, \omega)$, $\tilde{\phi}_i \in \widetilde{\text{Ham}}(M, \omega)$.

The case $m = 1$ follows from Lemma 14.7 which we apply to $\phi_i(U)$ in place of U . (We note that $\mathfrak{e}(\overline{U}; e; \mathfrak{b}) = \mathfrak{e}(\phi_i(\overline{U}); e; \mathfrak{b})$.)

Suppose the corollary is proved for $m - 1$. Applying the induction hypothesis to the case $m = 2$, we have

$$|\rho^{\mathfrak{b}}(\tilde{\phi}_1 \cdots \tilde{\phi}_m \tilde{\psi}; e) - \rho^{\mathfrak{b}}(\tilde{\phi}_1; e) - \rho^{\mathfrak{b}}(\tilde{\phi}_2 \cdots \tilde{\phi}_m \tilde{\psi}; e)| < 2\mathfrak{e}(\overline{U}; e; \mathfrak{b})$$

by Lemma 14.7. On the other hand, by the induction hypothesis we have

$$|\rho^{\mathfrak{b}}(\tilde{\phi}_2 \cdots \tilde{\phi}_m \tilde{\psi}; e) - \sum_{i=2}^m \rho^{\mathfrak{b}}(\tilde{\phi}_i; e) - \rho^{\mathfrak{b}}(\tilde{\psi}; e)| < 2(m-1)\mathfrak{e}(\overline{U}; e; \mathfrak{b}).$$

The inequality (14.11) follows. \square

We now prove the convergence of (14.2). Let $\tilde{\phi} \in \widetilde{\text{Ham}}_U(M, \omega)$. We have $\tilde{\phi}_i$ such that $\|\tilde{\phi}_i\|_U = 1$ and $\tilde{\phi} = \tilde{\phi}_1 \cdots \tilde{\phi}_m$. (Lemma 13.5.) We use Corollary 14.8 by putting $\tilde{\psi} = \overline{0}$ to obtain

$$|\rho^{\mathfrak{b}}(\tilde{\phi}^n; e) - n \sum_{i=1}^m \rho^{\mathfrak{b}}(\tilde{\phi}_i; e)| \leq 2mn\mathfrak{e}(\overline{U}; e; \mathfrak{b}). \quad (14.12)$$

We put

$$a_n = \rho^{\mathfrak{b}}(\tilde{\phi}^n; e) + 2mn\mathfrak{e}(\overline{U}; e; \mathfrak{b}) + nm|\sup\{\rho^{\mathfrak{b}}(\tilde{\phi}_i; e) \mid i = 1, \dots, m\}|.$$

(14.12) implies that $a_n \geq 0$. Theorem 7.8 (5) implies $a_n + a_{n'} \geq a_{n+n'}$. We recall the following:

Lemma 14.9. *If $a_n \geq 0$ and $a_n + a_{n'} \geq a_{n+n'}$, then $\lim_{n \rightarrow \infty} a_n/n$ converges.*

Proof. The following proof is taken from Problem 98 of p 17 [PS]. Since $a_{2^n}/2^n$ is nonincreasing, $\alpha = \liminf_{n \rightarrow \infty} a_n/n$ is a finite number. Let $\epsilon > 0$. We take any n_0 such that $|a_{n_0}/n_0| \leq \alpha + \epsilon$. If $n' = n_0 k + r$ with $r = 1, \dots, n_0 - 1$, then $a_{n'} = a_{n_0 k + r} \leq k a_{n_0} + a_r$. Therefore

$$\frac{a_{n'}}{n'} \leq \frac{a_{n_0}}{n_0} \frac{k n_0}{k n_0 + r} + \frac{a_r}{n'}.$$

Hence if n' is sufficiently large, we have $\alpha - \epsilon \leq a_{n'}/n' < \alpha + 2\epsilon$ as required. \square

We have thus proved that the limit

$$\text{vol}_\omega(M) \lim_{n \rightarrow +\infty} \frac{\rho^b(\tilde{\psi}^n; e)}{n}$$

exists.

The limit $\mu_e^b(\tilde{\psi})$ satisfies Definition 13.6 (2) by construction. Definition 13.6 (1) then follows from Theorem 7.8 (6). Definition 13.6 (4) follows from Theorem 7.8 (4).

We next prove the properties required in Definition 13.6 (3).

Lemma 14.10. *We have*

$$|\mu_e^b(\tilde{\phi}\tilde{\psi}) - \mu_e^b(\tilde{\phi}) - \mu_e^b(\tilde{\psi})| \leq 2\epsilon(\overline{U}; e; \mathbf{b}) \text{vol}_\omega(M) \min(2\|\tilde{\phi}\|_U - 1, 2\|\tilde{\psi}\|_U - 1). \quad (14.13)$$

Proof. We may assume without loss of generality that $\|\tilde{\phi}\|_U \leq \|\tilde{\psi}\|_U$. The proof is by induction on $m = \|\tilde{\phi}\|_U$.

We first consider the case $m = 1$. Since $\|\tilde{\psi}^j \tilde{\phi} \tilde{\psi}^{-j}\|_U = 1$, Corollary 14.8 and

$$(\tilde{\phi}\tilde{\psi})^k = \left(\prod_{j=0}^{k-1} \tilde{\psi}^j \tilde{\phi} \tilde{\psi}^{-j} \right) \tilde{\psi}^k$$

and $\rho^b(\tilde{\psi}^j \tilde{\phi} \tilde{\psi}^{-j}; e) = \rho^b(\tilde{\phi}; e)$ (Theorem 7.8 (4)) imply

$$|\rho^b((\tilde{\phi}\tilde{\psi})^k; e) - k\rho^b(\tilde{\phi}; e) - \rho^b(\tilde{\psi}^k; e)| \leq 2k\epsilon(\overline{U}; e; \mathbf{b}).$$

We use Corollary 14.8 again to derive

$$|\rho^b(\tilde{\phi}^k; e) - k\rho^b(\tilde{\phi}; e)| \leq 2k\epsilon(\overline{U}; e; \mathbf{b}).$$

Therefore

$$|\rho^b((\tilde{\phi}\tilde{\psi})^k; e) - \rho^b(\tilde{\phi}^k; e) - \rho^b(\tilde{\psi}^k; e)| \leq 4k\epsilon(\overline{U}; e; \mathbf{b}).$$

The case $m = 1$ of the lemma follows.

We assume that the lemma is proved for $m - 1$. We write $\tilde{\phi} = \tilde{\phi}_1 \tilde{\phi}_2$ with $\|\tilde{\phi}_1\|_U = m - 1$ and $\|\tilde{\phi}_2\|_U = 1$. Then by the induction hypothesis

$$|\mu_e^b(\tilde{\phi}\tilde{\psi}) - \mu_e^b(\tilde{\phi}_1) - \mu_e^b(\tilde{\phi}_2\tilde{\psi})| \leq 2\epsilon(\overline{U}; e; \mathbf{b}) \text{vol}_\omega(M) (2(m - 1) - 1).$$

The case $m = 1$ gives

$$|\mu_e^b(\tilde{\phi}_2\tilde{\psi}) - \mu_e^b(\tilde{\phi}_2) - \mu_e^b(\tilde{\psi})| \leq 2\epsilon(\overline{U}; e; \mathbf{b}) \text{vol}_\omega(M)$$

and

$$|\mu_e^b(\tilde{\phi}) - \mu_e^b(\tilde{\phi}_1) - \mu_e^b(\tilde{\phi}_2)| \leq 2\epsilon(\overline{U}; e; \mathbf{b}) \text{vol}_\omega(M).$$

(14.11) follows in the case of $\tilde{\phi}$. □

Lemma 14.10 implies

$$|\mu_e^b(\tilde{\phi}\tilde{\psi}) - \mu_e^b(\tilde{\phi}) - \mu_e^b(\tilde{\psi})| \leq 4\mathfrak{e}(\overline{U}; e; \mathfrak{b}) \text{vol}_\omega(M) \min(\|\tilde{\phi}\|_U, \|\tilde{\psi}\|_U). \quad (14.14)$$

Thus we have proved the property of Definition 13.6 (3).

Remark 14.11. We may take $K = 4\mathfrak{e}(\overline{U}; e; \mathfrak{b}) \text{vol}_\omega(M)$ for the constant in Definition 13.6 (3).

We next prove Definition 13.6 (5). Let $U \subset M$ be a displaceable open subset and $\tilde{\phi} \in \widetilde{\text{Ham}}_U(M, \omega)$. Let $\tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$ such that $\psi(U) \cap \overline{U} = \emptyset$. By Proposition 14.3 applied to $\tilde{\phi}^n$ we have

$$\rho^b(\tilde{\psi}\tilde{\phi}^n; e) = \rho^b(\tilde{\psi}; e) + \frac{n\text{Cal}_U(\tilde{\phi})}{\text{vol}_\omega(M)}.$$

Using this equality and Lemma 14.6, we obtain

$$\begin{aligned} \left| \rho^b(\tilde{\phi}^n; e) - \frac{n\text{Cal}_U(\tilde{\phi})}{\text{vol}_\omega(M)} \right| &= |\rho^b(\tilde{\phi}^n; e) + \rho^b(\tilde{\psi}; e) - \rho^b(\tilde{\psi}\tilde{\phi}^n; e)| \\ &\leq 2\mathfrak{e}(\overline{U}; e; \mathfrak{b}) < \infty. \end{aligned}$$

Then dividing this inequality by $\frac{n\text{Cal}_U(\tilde{\phi})}{\text{vol}_\omega(M)}$ and letting $n \rightarrow \infty$, we obtain $\mu^b(\tilde{\phi}) = \text{Cal}_U(\tilde{\phi})$. The proof of Theorem 14.1 (2) is complete.

We next turn to the proof of Theorem 14.1 (3), i.e., the functional $\zeta_e^b : C^0(M) \rightarrow \mathbb{R}$ is a partial symplectic quasi-state. For this purpose, we have only to consider autonomous smooth Hamiltonian F 's in the rest of the proof. Let F be a time *independent* Hamiltonian and take its normalization

$$\underline{F} = F - \frac{1}{\text{vol}_\omega(M)} \int_M F \omega^n. \quad (14.15)$$

Then

$$\rho^b(nF; e) + \int_M nF \omega^n = \rho^b(n\underline{F}; e) = \rho^b(\tilde{\psi}^n; e) \quad (14.16)$$

for $\tilde{\psi} = [\phi_F]$. Dividing this equation by n , we obtain

$$\frac{\rho^b(nF; e)}{n} + \frac{1}{\text{vol}_\omega(M)} \int_M F \omega^n = \frac{\rho^b(\tilde{\psi}^n; e)}{n}.$$

Therefore convergence of (14.4) follows from the convergence of (14.2). Thus $\zeta_e^b(F)$ is defined for $F \in C^\infty(M)$.

Definition 13.3 (1) is a consequence of Theorem 7.8 (6). We can extend ζ_e^b to $C^0(M)$ by the $F \in C^\infty(M)$ case of Definition 13.3 (1). The property of Definition 13.3 (1) in the case $F \in C^0(M)$ then follows for this extended ζ_e^b .

Since $\tilde{\psi}_{H/m}^m = \tilde{\psi}_H$ holds for autonomous Hamiltonian H , we can prove the property of Definition 13.3 (2) in the case $\lambda \in \mathbb{Q}_{\geq 0}$ by using Definition 13.6 (2) and (14.16). Then the case $\lambda \in \mathbb{R}_{\geq 0}$ follows from Definition 13.3 (1).

Definition 13.3 (4) is immediate from (14.16).

The property of Definition 13.3 (6) is a consequence of Theorem 7.8 (4).

Let us prove the property of Definition 13.3 (7). Suppose U is displaceable and the support of time independent Hamiltonian F is in U . We define U' as in (14.15). We take $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ such that $\phi(U) \cap \overline{U} = \emptyset$. By Proposition 14.3, we have:

$$\rho^b(\tilde{\phi}\tilde{\psi}_F^n; e) = \rho^b(\tilde{\phi}; e) + \frac{n\text{Cal}_U(\tilde{\psi}_F)}{\text{vol}_\omega(M)}. \quad (14.17)$$

(Here we use the fact that Cal_U is a homomorphism.)

By (14.11) we also have

$$|\rho^b(\tilde{\phi}\tilde{\psi}_F^n; e) - \rho^b(\tilde{\phi}; e) - \rho^b(\tilde{\psi}_F^n; e)| < 2\mathfrak{e}(\overline{U}; e; \mathfrak{b}).$$

Substituting (14.17) into this inequality, and then dividing by n and taking the limit, we obtain

$$\lim_{n \rightarrow \infty} \frac{\rho^b(\tilde{\psi}_F^n; e)}{n} = \frac{\text{Cal}_U(\tilde{\psi}_F)}{\text{vol}_\omega(M)} = \frac{1}{\text{vol}_\omega(M)} \int_M F \omega^n.$$

On the other hand, we have

$$\lim_{n \rightarrow \infty} \frac{\rho^b(\tilde{\psi}_F^n; e)}{n} = \lim_{n \rightarrow \infty} \frac{\rho^b(n\underline{F}; e)}{n} = -\zeta_e^b(\underline{F})$$

and hence

$$\zeta_e^b(F) = \zeta_e^b(\underline{F}) + \frac{1}{\text{vol}_\omega(M)} \int_M F \omega^n = 0.$$

We next prove the property of Definition 13.3 (3). Let $F_1 \leq F_2$. We put $H = F_1$ and $H' = F_2$ and apply the argument of the proof of Theorem 9.2 and obtain a chain map

$$\mathcal{P}_{(F_X; J_X), \#}^b : (CF(M; F_1; \Lambda^\perp), \partial_{(F_1, J_1)}^b) \rightarrow (CF(M; F_2; \Lambda^\perp), \partial_{(F_2, J_2)}^b).$$

Using $F_1 < F_2$ and Lemma 9.8 we have

$$\mathcal{P}_{(F_X; J_X), \#}^b(F^\lambda CF(M; F_1; \Lambda^\perp)) \subset F^\lambda CF(M; F_2; \Lambda^\perp).$$

Let $x \in F^\lambda CF(M; F_1; \Lambda^\perp)$ such that $[x] = [\mathcal{P}_{((F_1)_X, (J_1)_X), \#}^b(e)]$ and $|\lambda - \rho(F_1; e)| < \epsilon$. Then by Proposition 9.6 we have $[\mathcal{P}_{(F_X, J_X), \#}^b(x)] = [\mathcal{P}_{((F_2)_X, (J_2)_X), \#}^b(e)]$ and $\mathcal{P}_{(F_X, J_X), \#}^b(x) \in F^\lambda CF(M; F_2; \Lambda^\perp)$. Therefore $\rho^b(F_2; e) \leq \rho^b(F_1; e) + \epsilon$. It implies $\zeta_e^b(F_1) \leq \zeta_e^b(F_2)$, as required.

Next we prove the property of Definition 13.3 (5). By the assumption $\{F_1, F_2\} = 0$ we have

$$\tilde{\psi}_{F_1} \tilde{\psi}_{F_2} = \tilde{\psi}_{F_2} \tilde{\psi}_{F_1} = \tilde{\psi}_{F_1 + F_2}.$$

Therefore by Definition 13.6 (3) we have

$$\begin{aligned} & |\rho^b((\tilde{\psi}_{F_1} \tilde{\psi}_{F_2})^n; e) - \rho^b((\tilde{\psi}_{F_1})^n; e) - \rho^b((\tilde{\psi}_{F_2})^n; e)| \\ &= |\rho^b((\tilde{\psi}_{F_1})^n (\tilde{\psi}_{F_2})^n; e) - \rho^b((\tilde{\psi}_{F_1})^n; e) - \rho^b((\tilde{\psi}_{F_2})^n; e)| \leq K \|(\tilde{\psi}_{F_2})^n\|_U = K. \end{aligned}$$

Here U is a displaceable open set containing the support of F_2 . Therefore we have

$$\mu_e^b(\tilde{\psi}_{F_1} \tilde{\psi}_{F_2}) = \mu_e^b(\tilde{\psi}_{F_1}) + \mu_e^b(\tilde{\psi}_{F_2}) = \mu_e^b(\tilde{\psi}_{F_1}) + \text{Cal}_U(F_2).$$

(We use Definition 13.6 (5) in the second equality.)

$$\zeta_e^b(F_1 + F_2) = \zeta_e^b(F_1)$$

is now a consequence of (14.4). The triangle inequality for $\zeta_e^{\mathfrak{b}}$ follows the triangle inequality for the spectral invariant $\rho^{\mathfrak{b}}$, since $\zeta_e^{\mathfrak{b}}(F) = -\rho^{\mathfrak{b}}(F; e)$. The proof of Theorem 14.1 is now complete. \square

15. POINCARÉ DUALITY AND SPECTRAL INVARIANT

15.1. Statement of the result. Let $\pi : \Lambda^\downarrow \rightarrow \mathbb{C}$ be the projection to $\mathbb{C} \subset \Lambda^\downarrow$. Let

$$\langle \cdot, \cdot \rangle : \Omega(M) \otimes \Omega(M) \rightarrow \mathbb{C}$$

be the Poincaré duality pairing

$$\langle h_1, h_2 \rangle = \int_M h_1 \wedge h_2.$$

We extend the pairing to

$$\langle \cdot, \cdot \rangle : (\Omega(M) \widehat{\otimes} \Lambda^\downarrow) \otimes (\Omega(M) \widehat{\otimes} \Lambda^\downarrow) \rightarrow \Lambda^\downarrow$$

so that it becomes Λ^\downarrow -bilinear. We put

$$\Pi(a, b) = \pi(\langle a, b \rangle)$$

which induces a \mathbb{C} -bilinear pairing

$$\Pi : H(M; \Lambda^\downarrow) \otimes H(M; \Lambda^\downarrow) \rightarrow \mathbb{C}.$$

The main result of this section is:

Theorem 15.1. *Let $a \in H(M; \Lambda^\downarrow)$, $\mathfrak{b} \in H(M; \Lambda_0^\downarrow)$ and $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$. Then we have*

$$\rho^{\mathfrak{b}}(\tilde{\phi}; a) = -\inf_b \{\rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; b) \mid \Pi(a, b) \neq 0\}. \quad (15.1)$$

Remark 15.2. For the case $\mathfrak{b} = 0$, this theorem is due to Entov-Polterovich under the monotonicity assumption. (See [EP1] Lemma 2.2.) The assumptions on M which [EP1] imposed is removed and Theorem 15.1 itself is proved by Usher in [Us4].

15.2. Algebraic preliminary. In this section we prove some algebraic lemmas used in the proof of Theorem 15.1. A similar discussion was given by Usher in [Us3].

We work in the situation of Subsections 8.1. We put $G = \mathbb{R}$ in this subsection. Namely $C(G) = C(G') = C$. Note in this case we may take the basis e_i such that $\mathfrak{v}_q(e_i) = 0$. Let $\partial : C \rightarrow C$ be a boundary operator. We choose the standard basis e'_i, e''_i, e'''_i as in Subsection 8.1. Let D be another finite dimensional Λ^\downarrow vector space. We assume that there exists a Λ^\downarrow bilinear pairing

$$\langle \cdot, \cdot \rangle : C \times D \rightarrow \Lambda^\downarrow$$

that is perfect. (Namely it induces an isomorphism $C \rightarrow D^*$ to the dual space D^* of D .) Let $\{e_i^* \mid i = 1, \dots, N\}$ be the dual basis of $\{e_i \mid i = 1, \dots, N\}$. We use it to define the filtration $F^\lambda D$ in the same way as $F^\lambda C$. (We assume $\mathfrak{v}_q(e_i^*) = 0$.)

It is easy to see that if $x \in F^{\lambda_1} C$, $y \in F^{\lambda_2} D$ then

$$\langle x, y \rangle \in F^{\lambda_1 + \lambda_2} \Lambda^\downarrow. \quad (15.2)$$

We define $\partial^* : D \rightarrow D$ by

$$\langle x, \partial^* y \rangle = \langle \partial x, y \rangle.$$

It is easy to see that $\partial^* \circ \partial^* = 0$ and $\partial^*(F^\lambda D) \subset F^\lambda D$.

Definition 15.3. We call (D, ∂^*) the *filtered dual complex* of (C, ∂) .

We take a dual basis to $\{e'_i \mid i = 1, \dots, b\} \cup \{e''_i \mid i = 1, \dots, h\} \cup \{e'''_i \mid i = 1, \dots, b\}$. Namely we take $\{e'_{*i} \mid i = 1, \dots, b\} \cup \{e''_{*i} \mid i = 1, \dots, h\} \cup \{e'''_{*i} \mid i = 1, \dots, b\}$ such that

$$\langle e'_i, e'''_{*i} \rangle = 1, \quad \langle e''_i, e''_{*i} \rangle = 1, \quad \langle e'''_i, e'_{*i} \rangle = 1$$

and all the other pairings among the basis are zero. It is easy to see that $\{e'_{*i} \mid i = 1, \dots, b\}$ is a basis of $\text{Im } \partial^*$ and $\{e'_{*i} \mid i = 1, \dots, b\} \cup \{e''_{*i} \mid i = 1, \dots, h\}$ is a basis of $\text{Ker } \partial^*$.

In the same way as in (8.4) we have

$$\inf\{\mathbf{v}_q(x) \mid x \in \text{Ker } \partial^*, a = [x]\} = \mathbf{v}_q\left(\sum_{i=1}^h a_i e''_{*i}\right) \quad (15.3)$$

for $a \in H(D; \partial^*)$. We define $\mathbf{v}_q(a)$ for $a \in H(D; \partial^*)$ by the left hand side.

The pairing $\langle \cdot, \cdot \rangle$ induces a perfect Λ^\downarrow pairing between $H(C; \partial)$ and $H(D; \partial^*)$, which we also denote by $\langle \cdot, \cdot \rangle$. By (8.4) and (15.3) we have:

Lemma 15.4.

$$\mathbf{v}_q(a) = \sup\{\mathbf{v}_q(\langle a, b \rangle) \mid b \in H(F^0 D; \partial^*)\} \quad (15.4)$$

for $a \in H(C; \partial)$.

15.3. Duality between Floer homologies. Let H be a one periodic time dependent Hamiltonian on M . We assume that ψ_H is non-degenerate. We consider the chain complex $(CF(M, H; \Lambda^\downarrow), \partial_{(H, J)}^b)$ which is defined in Section 6.

Let $\{\gamma_i \mid i = 1, \dots, N\} = \text{Per}(H)$. We put

$$e_i = q^{\mathcal{A}_H([\gamma_i, w_i])}[\gamma_i, w_i] \in CF(M, H; \Lambda^\downarrow).$$

We note that e_i is independent of w_i . $\{e_i \mid 1, \dots, N\}$ is a Λ^\downarrow basis of $CF(M, H; \Lambda^\downarrow)$. It is easy to see that the filtration of $CF(M, H; \Lambda^\downarrow)$ defined as in Subsection 8.1 coincides with the filtration defined in Definition 2.4.

We define \tilde{H} by

$$\tilde{H}(t, x) = -H(1 - t, x). \quad (15.5)$$

We have $\phi_{\tilde{H}}^t = \phi_H^{1-t} \circ \phi_H^{-1}$. In particular, $\psi_{\tilde{H}} = (\psi_H)^{-1}$. Hence $\psi_{\tilde{H}}$ is also non-degenerate.

The main result of this subsection is as follows:

Proposition 15.5. *We can choose the perturbation etc. that are used in the definition of $(CF(M, \tilde{H}; \Lambda^\downarrow), \partial_{(\tilde{H}, J)}^b)$ such that there exists a perfect pairing*

$$\langle \cdot, \cdot \rangle : CF(M, H; \Lambda^\downarrow) \times CF(M, \tilde{H}; \Lambda^\downarrow) \rightarrow \Lambda^\downarrow$$

by which the filtered complex $(CF(M, \tilde{H}; \Lambda^\downarrow), \partial_{(\tilde{H}, \tilde{J})}^b)$ is identified with the dual filtered complex of $(CF(M, H; \Lambda^\downarrow), \partial_{(H, J)}^b)$.

Proof. Let $\gamma \in \text{Per}(H)$. It is then easy to see that

$$\tilde{\gamma}(t) = \gamma(1 - t) \in \text{Per}(\tilde{H}).$$

If $w : D^2 \rightarrow M$ satisfies $w|_{\partial D} = \gamma$, then $\tilde{w}(z) = w(\bar{z})$ satisfies $\tilde{w}|_{\partial D} = \tilde{\gamma}$. We have thus defined

$$\iota : \text{Crit}(\mathcal{A}_H) \rightarrow \text{Crit}(\mathcal{A}_{\tilde{H}}) \quad (15.6)$$

by $[\gamma, w] \mapsto [\tilde{\gamma}, \tilde{w}]$. It is easy to see

$$\mathcal{A}_H([\gamma, w]) + \mathcal{A}_{\tilde{H}}([\tilde{\gamma}, \tilde{w}]) = 0 \quad (15.7)$$

and

$$\int w^* \omega + \int \tilde{w}^* \omega = 0. \quad (15.8)$$

Let $(u; z_1^+, \dots, z_\ell^+) \in \mathring{\mathcal{M}}_\ell(H, J; [\gamma, w], [\gamma', w'])$. We define $\iota : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times S^1$ by $\iota(\tau, t) = (-\tau, 1 - t)$ and put

$$\tilde{u} = u \circ \iota. \quad (15.9)$$

It is easy to find that

$$(\tilde{u}; \tilde{z}_1^+, \dots, \tilde{z}_\ell^+) \in \mathring{\mathcal{M}}_\ell(\tilde{H}, \tilde{J}; [\tilde{\gamma}', \tilde{w}'], [\tilde{\gamma}, \tilde{w}]).$$

We thus defined a homeomorphism

$$\mathfrak{J} : \mathring{\mathcal{M}}_\ell(H, J; [\gamma, w], [\gamma', w']) \rightarrow \mathring{\mathcal{M}}_\ell(\tilde{H}, \tilde{J}; [\tilde{\gamma}', \tilde{w}'], [\tilde{\gamma}, \tilde{w}])$$

by

$$\mathfrak{J}(u; z_1^+, \dots, z_\ell^+) = (\tilde{u}; \tilde{z}_1^+, \dots, \tilde{z}_\ell^+).$$

We can extend it to their compactifications and then it becomes an isomorphism between spaces with Kuranishi structure:

$$\mathfrak{J} : \mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w']) \rightarrow \mathcal{M}_\ell(\tilde{H}, \tilde{J}; [\tilde{\gamma}', \tilde{w}'], [\tilde{\gamma}, \tilde{w}]).$$

We take a multisection of $\mathcal{M}_\ell(\tilde{H}, \tilde{J}; [\tilde{\gamma}', \tilde{w}'], [\tilde{\gamma}, \tilde{w}])$ so that it coincides with one for $\mathcal{M}_\ell(H, J; [\gamma, w], [\gamma', w'])$ by the above isomorphism. Then we have

$$\mathbf{n}_{(H, J); \ell}([\gamma, w], [\gamma', w'])(h_1, \dots, h_\ell) = \mathbf{n}_{(\tilde{H}, \tilde{J}); \ell}([\tilde{\gamma}', \tilde{w}'], [\tilde{\gamma}, \tilde{w}])(h_1, \dots, h_\ell),$$

where the left hand side is defined in (6.4). Therefore

$$\mathbf{n}_{(H, J)}^\flat([\gamma, w], [\gamma', w']) = \mathbf{n}_{(\tilde{H}, \tilde{J})}^\flat([\tilde{\gamma}', \tilde{w}'], [\tilde{\gamma}, \tilde{w}]). \quad (15.10)$$

Definition 15.6. Let $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$, $[\tilde{\gamma}, \tilde{w}] \in \text{Crit}(\mathcal{A}_{\tilde{H}})$. We define

$$\langle [\gamma, w], [\tilde{\gamma}', \tilde{w}'] \rangle = \begin{cases} 0 & \text{if } \gamma \neq \gamma', \\ q^{-(w \cap \omega + \tilde{w}' \cap \omega)} & \text{if } \gamma = \gamma'. \end{cases} \quad (15.11)$$

We can extend (15.11) to a Λ^\perp bilinear pairing

$$\langle \cdot, \cdot \rangle : CF(M, H; \Lambda^\perp) \times CF(M, \tilde{H}; \Lambda^\perp) \rightarrow \Lambda^\perp,$$

which becomes a perfect pairing.

By (15.8) we have

$$\langle [\gamma, w], [\tilde{\gamma}, \tilde{w}] \rangle = 1. \quad (15.12)$$

Lemma 15.7.

$$\langle \partial_{(H, J)}^\flat([\gamma_1, w_1]), [\tilde{\gamma}_2, \tilde{w}_2] \rangle = \langle [\gamma_1, w_1], \partial_{(\tilde{H}, \tilde{J})}^\flat([\tilde{\gamma}_2, \tilde{w}_2]) \rangle. \quad (15.13)$$

Proof. By definition the left hand side is

$$\begin{aligned} & \sum_{w'_2} \mathbf{n}_{(H,J)}^b([\gamma_1, w_1], [\gamma_2, w'_2]) q^{-(w'_2 \cap \omega + \tilde{w}_2 \cap \omega)} \\ &= \sum_{\alpha \in H_2(M; \mathbb{Z})} \mathbf{n}_{(H,J)}^b([\gamma_1, w_1], [\gamma_2, w_2 + \alpha]) q^{-\alpha \cap \omega}. \end{aligned}$$

On the other hand, the right hand side is

$$\begin{aligned} & \sum_{\tilde{w}'_1} \mathbf{n}_{(\tilde{H}, \tilde{J})}^b([\tilde{\gamma}_2, \tilde{w}_2], [\tilde{\gamma}_1, \tilde{w}'_1]) q^{-(w_1 \cap \omega + \tilde{w}'_1 \cap \omega)} \\ &= \sum_{\alpha \in H_2(M; \mathbb{Z})} \mathbf{n}_{(\tilde{H}, \tilde{J})}^b([\tilde{\gamma}_2, \tilde{w}_2], [\tilde{\gamma}_1, \widetilde{(w_1 - \alpha)}]) q^{-\alpha \cap \omega}. \end{aligned}$$

By (15.10) this is equal to

$$\mathbf{n}_{(H,J)}^b([\gamma_1, w_1 - \alpha], [\gamma_2, w_2]) q^{-\alpha \cap \omega}.$$

Since

$$\mathbf{n}_{(H,J)}^b([\gamma_1, w_1 - \alpha], [\gamma_2, w_2]) = \mathbf{n}_{(H,J)}^b([\gamma_1, w_1], [\gamma_2, w_2 + \alpha]),$$

the lemma follows. \square

(15.12) and (15.13) imply the proposition. \square

15.4. Duality and Piunikhin isomorphism. In this subsection we prove:

Theorem 15.8. *For $a, a' \in H^*(M; \Lambda^\perp)$ we denote by $a^\flat, (a')^\flat$ the homology classes Poincaré dual to a, a' respectively. (See Notations and Conventions (17).) Then we have*

$$\langle \mathcal{P}_{(H_\chi, J_\chi), *}^b(a^\flat), \mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), *}^b((a')^\flat) \rangle = \langle a, a' \rangle. \quad (15.14)$$

Proof. We consider two chain maps $: (\Omega(M) \hat{\otimes} \Lambda^\perp) \otimes (\Omega(M) \hat{\otimes} \Lambda^\perp) \rightarrow \Lambda^\perp$

$$h \otimes h' \mapsto \int_M h \wedge h' \quad (15.15)$$

and

$$h \otimes h' \mapsto \langle \mathcal{P}_{(H_\chi, J_\chi), \#}^b(h), \mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), \#}^b(h') \rangle. \quad (15.16)$$

Here we regard Λ^\perp as a chain complex with trivial boundary operator. To prove Theorem 15.8 it suffices to show that (15.15) is chain homotopic to (15.16). For this purpose, we will use the following parameterized moduli space

$$\mathcal{M}_\ell(\text{para} : H_\chi, J_\chi; *, *, C) = \bigcup_{S \geq 0} \{S\} \times \mathcal{M}_\ell(H_\chi^S, J_\chi^S; *, *, C)$$

equipped with Kuranishi structure and multisection that is compatible at the boundary. We refer readers to Definition 26.6 in Section 26 for the precise description of $\mathcal{M}_\ell(\text{para} : H_\chi, J_\chi; *, *, C)$ defined in (26.15).

We denote $\tilde{\chi} = \tilde{\chi}(\tau) = \chi(-\tau)$. Some boundary component of $\mathcal{M}_\ell(\text{para} : H_\chi, J_\chi; *, *, C)$ in (26.16) will contain a direct factor of the type $\mathcal{M}_{\# \mathbb{L}_2}(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ whose definition is given in Definition 26.2. We consider the map

$$\mathfrak{J} : \mathcal{M}_\ell(\tilde{H}_\chi, \tilde{J}_\chi; *, [\tilde{\gamma}, \tilde{w}]) \rightarrow \mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *) \quad (15.17)$$

defined by

$$\mathfrak{J}((u; z_1^+, \dots, z_\ell^+)) = (\tilde{u}; \tilde{z}_1^+, \dots, \tilde{z}_\ell^+),$$

where the right hand side is defined as in (15.9). The homomorphism (15.17) is extended to an isomorphism of spaces with Kuranishi structures.

Recall that when we considered $\mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi)}^b$, we made a choice of a multisection on $\mathcal{M}_\ell(\tilde{H}_\chi, \tilde{J}_\chi; *, [\tilde{\gamma}, \tilde{w}])$. This multisection induces a multisection on $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ via the isomorphism (15.17).

We equip $\mathcal{M}_\ell(\text{para} : H_\chi, J_\chi; *, *, C)$ with a system of multisections that is compatible at the boundary with respect to this choice of multisection on the direct factor $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ appearing in (26.16).

Remark 15.9. On the other hand, when we will define $\mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^b$ in Section 26, we take another family of multisections on $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$. This is *different* from the multisection defined above.

Now let h, h' be differential forms on M . We define

$$\begin{aligned} & \overline{\mathcal{H}}_{(H_\chi, J_\chi)}^b(h, h') \\ &= \sum_C \sum_{\ell=0}^{\infty} \frac{\exp(C \cap \mathfrak{b}_2)}{\ell!} q^{-C \cap \omega} \int_{\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)} ev_{+\infty}^* h \wedge ev_{-\infty}^* h' \wedge ev^* \underbrace{(\mathfrak{b}_+, \dots, \mathfrak{b}_+)}_{\ell}, \end{aligned} \quad (15.18)$$

where \mathfrak{b}_2 is the summand in the decomposition $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_2 + \mathfrak{b}_+$ as before and we use the above chosen multisection on $\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$ to define an integration on it. The formula (15.18) defines a map

$$\overline{\mathcal{H}}_{(H_\chi, J_\chi)}^b : (\Omega(M) \widehat{\otimes} \Lambda^\downarrow) \otimes (\Omega(M) \widehat{\otimes} \Lambda^\downarrow) \rightarrow \Lambda^\downarrow.$$

It follows from Lemma 26.8 (3) that $\overline{\mathcal{H}}_{(H_\chi, J_\chi)}^b$ is a chain homotopy between (15.15) and (15.16). The proof of Theorem 15.8 is complete. \square

15.5. Proof of Theorem 15.1. Now we prove Theorem 15.1. Once Theorem 15.8 is established, the proof is the same as [EP2]. It suffices to prove it in the case when $\tilde{\phi}$ is nondegenerate. We take H such that $\tilde{\phi} = \tilde{\psi}_H$. Let $a \in H(M; \Lambda^\downarrow)$ and $\epsilon > 0$. By Lemma 15.4, we have $b' \in H(M; \Lambda^\downarrow)$ such that

$$\mathfrak{v}_q \left(\left\langle \mathcal{P}_{(H_\chi, J_\chi), *}^b(a^b), \mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), *}^b((b')^b) \right\rangle \right) \geq \rho^b(H; a) - \epsilon \quad (15.19)$$

and

$$\mathfrak{v}_q(\mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), *}^b((b')^b)) \leq 0.$$

Let λ be the left hand side of (15.19). Then

$$0 = \mathfrak{v}_q \left(\left\langle \mathcal{P}_{(H_\chi, J_\chi), *}^b(a^b), \mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), *}^b(q^{-\lambda}(b')^b) \right\rangle \right) = \mathfrak{v}_q \left(\langle a, q^{-\lambda} b' \rangle \right).$$

(We use Theorem 15.8 here.) We put $b = q^{-\lambda} b'$. Then by definition

$$\Pi(a, b) \neq 0.$$

Thus, since $\mathfrak{v}_q(\mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), *}^b(b^b)) = -\lambda + \mathfrak{v}_q(\mathcal{P}_{(\tilde{H}_\chi, \tilde{J}_\chi), *}^b((b')^b)) \leq -\lambda$, we have

$$\rho^b(H; a) - \epsilon \leq \lambda \leq -\inf\{\rho^b(\tilde{\psi}_H^{-1}; b) \mid \Pi(a, b) \neq 0\}.$$

Hence

$$\rho^b(\tilde{\psi}_H; a) \leq -\inf\{\rho^b(\tilde{\psi}_H^{-1}; b) \mid \Pi(a, b) \neq 0\}. \quad (15.20)$$

On the other hand, if $\Pi(a, b) \neq 0$ then

$$\mathfrak{v}_q(\langle a, b \rangle) \geq 0.$$

It implies

$$\mathfrak{v}_q \left(\left\langle \mathcal{P}_{(H_X, J_X), *}^{\mathfrak{b}}(a^{\mathfrak{b}}), \mathcal{P}_{(\tilde{H}_X, \tilde{J}_X), *}^{\mathfrak{b}}(b^{\mathfrak{b}}) \right\rangle \right) \geq 0.$$

Hence

$$\mathfrak{v}_q(\mathcal{P}_{(H_X, J_X), *}^{\mathfrak{b}}(a^{\mathfrak{b}})) + \mathfrak{v}_q(\mathcal{P}_{(\tilde{H}_X, \tilde{J}_X), *}^{\mathfrak{b}}(b^{\mathfrak{b}})) \geq 0.$$

Therefore

$$\rho^{\mathfrak{b}}(\tilde{\psi}_H; a) \geq -\inf\{\rho^{\mathfrak{b}}(\tilde{\psi}_H^{-1}; b) \mid \Pi(a, b) \neq 0\}. \quad (15.21)$$

(15.20) and (15.21) imply Theorem 15.1. \square

16. CONSTRUCTION OF QUASIMORPHISMS VIA SPECTRAL INVARIANT WITH BULK

The next definition is due to Entov-Polterovich [EP1] Section 1.1.

Definition 16.1. A function $\mu : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$ is called a *homogeneous Calabi quasimorphism* if the following three conditions are satisfied.

- (1) It is a quasimorphism. Namely there exists a constant C such that for any $\tilde{\phi}, \tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$ we have

$$|\mu(\tilde{\phi}\tilde{\psi}) - \mu(\tilde{\phi}) - \mu(\tilde{\psi})| < C,$$

where C is independent of $\tilde{\phi}, \tilde{\psi}$.

- (2) If $\tilde{\phi} \in \widetilde{\text{Ham}}_U(M, \omega)$ and U is a displaceable open subset of M , then we have

$$\mu(\tilde{\phi}) = \text{Cal}_U(\tilde{\phi}).$$

- (3) It is homogeneous. Namely

$$\mu(\tilde{\phi}^n) = n\mu(\tilde{\phi})$$

for $n \in \mathbb{Z}$.

Remark 16.2. We note that we have the canonical homomorphism $\widetilde{\text{Ham}}_U(M, \omega) \rightarrow \widetilde{\text{Ham}}(M, \omega)$. We use this homomorphism to make sense out of the left hand side of the identity (2) above.

The following is the analog to Theorem 3.1 [EP1] whose proof is essentially the same once Theorem 15.1 is at our disposal.

Theorem 16.3. Let $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0^{\downarrow})$. Suppose that there is a ring isomorphism

$$QH_{\mathfrak{b}}^*(M; \Lambda^{\downarrow}) \cong \Lambda^{\downarrow} \times Q$$

and let $e \in QH_{\mathfrak{b}}^*(M; \Lambda^{\downarrow})$ be the idempotent corresponding to the unit of the first factor of the right hand side. Then the function

$$\mu_e^{\mathfrak{b}} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$$

is a homogeneous Calabi quasimorphism.

Remark 16.4. An observation by McDuff is that a sufficient condition for the existence of Calabi quasimorphism is an existence of a direct product factor of a quantum cohomology that is a field. [EP1] used quantum homology over $\Lambda^\downarrow(\mathbb{Q})$, that is the set of $\sum a_i q^{\lambda_i}$ with $a_i \in \mathbb{Q}$. Here we use the (downward) universal Novikov ring Λ^\downarrow , where $a_i \in \mathbb{C}$. Since Λ^\downarrow is an algebraically closed field (see [FOOO2] Appendix A) and cohomology ring is finite dimensional, the direct product factor of a quantum cohomology is isomorphic to Λ^\downarrow , if it is a field. So our assumption of Theorem 16.3 is equivalent to McDuff's in case $\mathfrak{b} = 0$.

Proof. Let e and \mathfrak{b} as in Theorem 16.3. We first prove the property (1) of Definition 16.1. We begin with the following lemma.

Lemma 16.5.

$$\rho^{\mathfrak{b}}(\tilde{\phi}; e) \leq 3\mathfrak{v}_q(e) - \rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; e).$$

Proof. Let $b \in H(M; \Lambda^\downarrow)$ such that $\Pi(e, b) \neq 0$. Such a b exists by the nondegeneracy of the Poincaré pairing. We write $b = (b_1, b_2)$ according to the decomposition $QH_b^*(M; \Lambda^\downarrow) \cong \Lambda^\downarrow \times Q$. Using the Frobenius property of quantum cohomology (see, for example, [Man]) we obtain

$$\langle e, b \rangle = \langle e \cup^{\mathfrak{b}} e, b \rangle = \langle e, e \cup^{\mathfrak{b}} b \rangle = \langle e, b_1 \rangle.$$

Therefore $\Pi(e, b_1) = \Pi(e, b) \neq 0$.

Sublemma 16.6. $\mathfrak{v}_q(b_1) \geq 0$.

Proof. We have $b_1 = xe$ for some $x \in \Lambda^\downarrow$. We decompose $e = \sum_{d=0}^{2n} e_d$ with $e_d \in H^d(M; \mathbb{C}) \otimes \Lambda^\downarrow$. We denote by $\mathbf{1} \in H^0(M; \mathbb{C})$ the unit of the cohomology ring. Then

$$\begin{aligned} \Pi(e, b_1) &= \pi(\langle e, b_1 \rangle) = \pi(\langle e, xe \rangle) = \pi(\langle e \cup^{\mathfrak{b}} e, x\mathbf{1} \rangle) \\ &= \pi(\langle e, x\mathbf{1} \rangle) = \pi(\langle xe, \mathbf{1} \rangle) = \pi(\langle b_1, \mathbf{1} \rangle). \end{aligned}$$

Therefore $\Pi(e, b_1) \neq 0$ implies $\mathfrak{v}_q(\langle b_1, \mathbf{1} \rangle) > 0$. Since $\mathfrak{v}_q(b_1) \geq \mathfrak{v}_q(\langle b_1, \mathbf{1} \rangle)$, we obtain $\mathfrak{v}_q(b_1) \geq 0$ as required. \square

Let $xe = b_1$ and $x \in \Lambda^\downarrow$ as above. Then

$$\mathfrak{v}_q(x) + \mathfrak{v}_q(e) = \mathfrak{v}_q(b_1) \geq 0.$$

Since $b_1^{-1} = x^{-1}e$, we get

$$\mathfrak{v}_q(b_1^{-1}) = -\mathfrak{v}_q(x) + \mathfrak{v}_q(e) = (\mathfrak{v}_q(e) - \mathfrak{v}_q(b_1)) + \mathfrak{v}_q(e) \leq 2\mathfrak{v}_q(e).$$

Therefore

$$\begin{aligned} \rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; b) &\geq \rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; b_1) - \rho^{\mathfrak{b}}(\underline{0}; e) \\ &\geq \rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; e) - \rho^{\mathfrak{b}}(\underline{0}; b_1^{-1}) - \rho^{\mathfrak{b}}(\underline{0}; e) = \rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; e) - \mathfrak{v}_q(b_1^{-1}) - \mathfrak{v}_q(e) \\ &\geq \rho^{\mathfrak{b}}(\tilde{\phi}^{-1}; e) - 3\mathfrak{v}_q(e). \end{aligned}$$

Here we use the identity $b_1 \cup^{\mathfrak{b}} b_1^{-1} = e$ in the second inequality. Lemma 16.5 now follows from Theorem 15.1. \square

Corollary 16.7.

$$\rho^{\mathfrak{b}}(\tilde{\phi}; e) + \rho^{\mathfrak{b}}(\tilde{\psi}; e) \geq \rho^{\mathfrak{b}}(\tilde{\psi}\tilde{\phi}; e) \geq \rho^{\mathfrak{b}}(\tilde{\phi}; e) + \rho^{\mathfrak{b}}(\tilde{\psi}; e) - 3\mathfrak{v}_q(e).$$

Proof. The first inequality is a consequence of Theorem 7.8 (5).

We have

$$\rho^b(\tilde{\psi}\tilde{\phi}; e) \geq \rho^b(\tilde{\psi}; e) - \rho^b(\tilde{\phi}^{-1}; e) \geq \rho^b(\tilde{\psi}; e) + \rho^b(\tilde{\phi}; e) - 3\mathfrak{v}_q(e).$$

Here the first inequality follows from Theorem 7.8 (5) and the second inequality follows from Lemma 16.5. \square

We use Corollary 16.7 inductively to show

$$\left| \rho^b(\tilde{\phi}_1 \cdots \tilde{\phi}_k; e) - \sum_{i=1}^k \rho^b(\tilde{\phi}_i; e) \right| \leq 3k\mathfrak{v}_q(e). \quad (16.1)$$

Therefore

$$\begin{aligned} \left| \rho^b((\tilde{\phi}\tilde{\psi})^n; e) - n\rho^b(\tilde{\phi}; e) - n\rho^b(\tilde{\psi}; e) \right| &\leq 6n\mathfrak{v}_q(e). \\ \left| \rho^b(\tilde{\phi}^n; e) - n\rho^b(\tilde{\phi}; e) \right| &\leq 3n\mathfrak{v}_q(e). \\ \left| \rho^b(\tilde{\psi}^n; e) - n\rho^b(\tilde{\psi}; e) \right| &\leq 3n\mathfrak{v}_q(e). \end{aligned}$$

Hence

$$\left| \rho^b((\tilde{\phi}\tilde{\psi})^n; e) - \rho^b(\tilde{\phi}^n; e) - \rho^b(\tilde{\psi}^n; e) \right| \leq 12n\mathfrak{v}_q(e).$$

It implies

$$|\mu_e^b(\tilde{\phi}\tilde{\psi}) - \mu_e^b(\tilde{\phi}) - \mu_e^b(\tilde{\psi})| \leq 12\mathfrak{v}_q(e).$$

Thus, μ_e^b is a quasimorphism.

Remark 16.8. (1) The constant C in Definition 16.1 can be taken to be $12\mathfrak{v}_q(e)$ for the quasimorphism in Theorem 16.3.

(2) Our proof of Lemma 16.5 is slightly simpler than [EP1] Lemma 3.2, since we may assume that the field which is a direct factor of quantum cohomology is Λ^\downarrow and so we do not need a result from general non-Archimedean geometry which is quoted in [EP1]. By the same reason we obtain an explicit bound.

Definition 16.1 (2) follows from Theorem 13.6 (5).

The homogeneity of μ_e^b follows from

$$\rho^b(\mathbb{Q}; e) \leq \rho^b(\tilde{\phi}^n; e) + \rho^b(\tilde{\phi}^{-n}; e) \leq 3\mathfrak{v}_q(e)$$

and Definition 13.6 (2). The proof of Theorem 16.3 is complete. \square

Theorem 1.3 is immediate from Theorem 16.3. \square

Part 4. Spectral invariants and Lagrangian Floer theory

The purpose of this chapter is to prove Theorem 1.7. The proof is based on open-closed Gromov-Witten theory developed in [FOOO1] Section 3.8, which induces a map from the quantum cohomology of the ambient symplectic manifold to the Hochschild cohomology of A_∞ algebra (or more generally that of Fukaya category of (M, ω)). A part of this map is defined in [FOOO1]. See also [FOOO6] Section 31. For our purpose, we need only a small portion thereof, that is, the part constructed in [FOOO1] Theorem 3.8.62 to which we restrict ourselves in this paper, except in Section 25.1.

The main new part of the proof is the construction of a map from Floer homology of periodic Hamiltonians to the Floer cohomology of Lagrangian submanifold, through which the map from quantum cohomology to Floer cohomology of Lagrangian submanifold factors (Definition 18.16 and Proposition 18.21). We also study its properties, especially those related to the filtration.

In Chapters 4 and 5, we fix a compatible almost complex structure J that is t -independent.

17. OPERATOR \mathbf{q} ; REVIEW

In this section, we review a part of the results of Section 3.8 [FOOO1].

Let (M, ω) be a compact symplectic manifold and L its relatively spin Lagrangian submanifold. We consider smooth differential forms on M . Note in [FOOO1, FOOO3] we used smooth singular chains instead of differential forms to represent cohomology classes on M . In this paper we use differential forms because we use them in the discussion of Floer homology in Chapter 2. The construction of the operator \mathbf{q} in this section is a minor modification of the one given in Section 3.8 [FOOO1] where smooth singular chains on M are used.

We will introduce a family of operators denoted by

$$\mathbf{q}_{\ell, k; \beta} : E_\ell(\Omega(M)[2]) \otimes B_k(\Omega(L)[1]) \rightarrow \Omega(L)[1]. \quad (17.1)$$

Explanation of the various notations appearing in (17.1) is in order. β is an element of the image of $\pi_2(M, L) \rightarrow H_2(M, L; \mathbb{Z})$ and $C[i]$ is the degree shift of a \mathbb{Z} graded \mathbb{C} -vector space C by i defined by $(C[i])^d = C^{d+i}$. We recall from Notations and Conventions (19)–(20) that $E_\ell C$ is the quotient of $B_\ell C = \underbrace{C \otimes \cdots \otimes C}_{\ell \text{ times}}$ by the

symmetric group action. The map (17.1) is a \mathbb{C} -linear map of degree $1 - \mu(\beta)$ here μ is the Maslov index.

We next describe the main properties of $\mathbf{q}_{\ell, k; \beta}$. Recall from Notations and Conventions (19)–(20) again that $BC = \bigoplus_{k=0}^\infty B_k C$ and $EC = \bigoplus_{\ell=0}^\infty E_\ell C$ have structures of coassociative coalgebras with coproducts Δ . We also consider a map $\Delta^{n-1} : BC \rightarrow (BC)^{\otimes n}$ or $EC \rightarrow (EC)^{\otimes n}$ defined by

$$\Delta^{n-1} = (\Delta \otimes \underbrace{id \otimes \cdots \otimes id}_{n-2}) \circ (\Delta \otimes \underbrace{id \otimes \cdots \otimes id}_{n-3}) \circ \cdots \circ \Delta.$$

An element $\mathbf{x} \in BC$ can be expressed as

$$\Delta^{n-1}(\mathbf{x}) = \sum_c \mathbf{x}_c^{n;1} \otimes \cdots \otimes \mathbf{x}_c^{n;n} \quad (17.2)$$

where c runs over some index set depending on \mathbf{x} . Here we note that by Notations and Conventions (21) we always use the coproducts Δ_{decon} on $B(\Omega(L)[1])$ and

Δ_{shuff} on $E(\Omega(M)[2])$, respectively. Thus for $\mathbf{x} \in B(\Omega(L)[1])$ the equation (17.2) expresses the decomposition of $\Delta_{\text{decon}}^{n-1}(\mathbf{x})$, while for $\mathbf{y} \in E(\Omega(M)[2])$ the equation (17.2) expresses the decomposition of $\Delta_{\text{shuff}}^{n-1}(\mathbf{y})$. For an element $\mathbf{x} = x_1 \otimes \cdots \otimes x_k \in B_k(\Omega(L)[1])$ we put the shifted degree $\deg' x_i = \deg x_i - 1$ and $\deg' \mathbf{x} = \sum \deg' x_i = \deg \mathbf{x} - k$. (Recall $\deg x_i$ is the cohomological degree of x_i before shifted.) The next result is the de Rham version of Theorem 3.8.32 [FOOO1].

Theorem 17.1. *The operators $\mathbf{q}_{\beta;\ell,k}$ have the following properties:*

- (1) *For each β and $\mathbf{x} \in B_k(\Omega(L)[1])$, $\mathbf{y} \in E_k(\Omega(M)[2])$, we have the following:*

$$0 = \sum_{\beta_1+\beta_2=\beta} \sum_{c_1, c_2} (-1)^* \mathbf{q}_{\beta_1}(\mathbf{y}_{c_1}^{2;1}; \mathbf{x}_{c_2}^{3;1} \otimes \mathbf{q}_{\beta_2}(\mathbf{y}_{c_1}^{2;2}; \mathbf{x}_{c_2}^{3;2}) \otimes \mathbf{x}_{c_2}^{3;3}) \quad (17.3)$$

where $*$ = $\deg' \mathbf{x}_{c_2}^{3;1} + \deg' \mathbf{x}_{c_2}^{3;1} \deg \mathbf{y}_{c_1}^{2;2} + \deg \mathbf{y}_{c_1}^{2;1}$. In (17.3) and hereafter, we write $\mathbf{q}_{\beta}(\mathbf{y}; \mathbf{x})$ in place of $\mathbf{q}_{\ell,k;\beta}(\mathbf{y}; \mathbf{x})$ if $\mathbf{y} \in E_{\ell}(\Omega(M)[2])$, $\mathbf{x} \in B_k(\Omega(L)[1])$. We use notation (17.2) in (17.3).

- (2) *If $1 \in E_0(\Omega(M)[2])$ and $\mathbf{x} \in B_k(\Omega(L)[1])$ then*

$$\mathbf{q}_{0,k;\beta}(1; \mathbf{x}) = \mathbf{m}_{k;\beta}(\mathbf{x}). \quad (17.4)$$

Here $\mathbf{m}_{k;\beta}$ is the filtered A_{∞} structure on $\Omega(L)$.

- (3) *Let \mathbf{e} be the 0 form (function) on L which is 1 everywhere. Let $\mathbf{x}_i \in B(\Omega(L)[1])$ and we put $\mathbf{x} = \mathbf{x}_1 \otimes \mathbf{e} \otimes \mathbf{x}_2 \in B(\Omega(L)[1])$. Then*

$$\mathbf{q}_{\beta}(\mathbf{y}; \mathbf{x}) = 0 \quad (17.5)$$

except the following case.

$$\mathbf{q}_{\beta_0}(1; \mathbf{e} \otimes x) = (-1)^{\deg x} \mathbf{q}_{\beta_0}(1; x \otimes \mathbf{e}) = x, \quad (17.6)$$

where $\beta_0 = 0 \in H_2(M, L; \mathbb{Z})$ and $x \in \Omega(L)[1] = B_1(\Omega(L)[1])$. Note 1 in (17.6) is $1 \in E_0(\Omega(M)[2])$.

The singular homology version of Theorem 17.1 is proved in Sections 3.8 and 7.4 of [FOOO1]. The version where we use de Rham cohomology for L and cycles (smooth submanifolds) on M is in Section 6 of [FOOO2] for the case when M is a toric manifold and L is a Lagrangian torus fiber of M .

Since we use the details of the construction in the proof of Theorem 18.8 later in Section 18, we explain the construction of the relevant operators and the main ideas used in the proof of Theorem 17.1, although it is a straightforward modification of the construction of [FOOO1, FOOO2].

Definition 17.2. We denote by $\overset{\circ}{\mathcal{M}}_{k+1;\ell}(L; \beta)$ the set of all \sim equivalence classes of triples $(u; z_1^+, \dots, z_{\ell}^+; z_0, \dots, z_k)$ satisfying the following:

- (1) $u : (D^2, \partial D^2) \rightarrow (M, L)$ is a pseudo-holomorphic map such that $u(\partial D^2) \subset L$.
- (2) z_1^+, \dots, z_{ℓ}^+ are points in the interior of D^2 which are mutually distinct.
- (3) z_0, \dots, z_k are points on the boundary ∂D^2 of D^2 . They are mutually distinct. z_0, \dots, z_k respects the counterclockwise cyclic order on ∂D^2 .
- (4) The homology class $u_*([D^2, \partial D^2])$ is $\beta \in H_2(M, L; \mathbb{Z})$.

We say that $(u; z_1^+, \dots, z_{\ell}^+; z_0, \dots, z_k) \sim (u'; z_1'^+, \dots, z_{\ell}'^+; z_0', \dots, z_k')$ if there exists a biholomorphic map $v : D^2 \rightarrow D^2$ such that

$$u' \circ v = u, \quad v(z_i^+) = z_i'^+, \quad v(z_i) = z_i'.$$

We define an evaluation map

$$(\text{ev}, \text{ev}^\partial) = (\text{ev}_1, \dots, \text{ev}_\ell; \text{ev}_0^\partial, \dots, \text{ev}_k^\partial) : \mathring{\mathcal{M}}_{k+1;\ell}(L; \beta) \rightarrow M^\ell \times L^{k+1}$$

by

$$\text{ev}_i([u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k]) = u(z_i^+), \quad \text{ev}_i^\partial([u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k]) = u(z_i).$$

- Proposition 17.3.** (1) *The moduli space $\mathring{\mathcal{M}}_{k+1;\ell}(L; \beta)$ has a compactification $\mathcal{M}_{k+1;\ell}(L; \beta)$ that is Hausdorff.*
 (2) *The space $\mathcal{M}_{k+1;\ell}(L; \beta)$ has an orientable Kuranishi structure with corners.*
 (3) *The boundary of $\mathcal{M}_{k+1;\ell}(L; \beta)$ in the sense of Kuranishi structure is described by the following fiber product over L .*

$$\partial \mathcal{M}_{k+1;\ell}(L; \beta) = \bigcup \mathcal{M}_{k_1+1;\#\mathbb{L}_1}(L; \beta_1)_{\text{ev}_0^\partial} \times_{\text{ev}_i^\partial} \mathcal{M}_{k_2+1;\#\mathbb{L}_2}(L; \beta_2), \quad (17.7)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ with $k_1 + k_2 = k$ and $\beta_1, \beta_2 \in H_2(M, L; \mathbb{Z})$ with $\beta_1 + \beta_2 = \beta$.

- (4) *There exists a map $\mu_L : H_2(M, L; \mathbb{Z}) \rightarrow 2\mathbb{Z}$, Maslov index, such that the (virtual) dimension satisfies the following equality (17.8).*

$$\dim \mathcal{M}_{k+1;\ell}(L; \beta) = n + \mu_L(\beta) + k - 2 + 2\ell. \quad (17.8)$$

- (5) *We can define orientations of $\mathcal{M}_{k+1;\ell}(L; \beta)$ so that (3) above is compatible with this orientation in the sense of Proposition 8.3.3 in [FOOO1].*
 (6) *The evaluation map is extended to the compactification so that it is compatible with (17.7).*
 (7) *ev_0^∂ is weakly submersive.*
 (8) *The Kuranishi structure is compatible with the forgetful map of the boundary marked points.*
 (9) *The Kuranishi structure is invariant under the permutation of interior marked points.*
 (10) *The Kuranishi structure is invariant under the cyclic permutation of the boundary marked points.*

Proposition 17.3 (1) - (7) is proved in [FOOO1] Propositions 7.1.1, 7.1.2, (that is the case $\ell = 0$. The case $\ell \neq 0$ is the same). The Kuranishi structure satisfying the additional properties (8), (9), (10) is constructed in Corollary 3.1 [Fu3]. We refer [Fu3] Definition 3.1 for the precise meaning of the statement (8).

Lemma 17.4. *There exists a system of continuous families of multisections on the moduli spaces $\mathcal{M}_{k+1;\ell}(L; \beta)$ such that the following holds:*

- (1) *It is transversal to zero.*
- (2) *It is compatible with the description of the boundary in Proposition 17.3 (3) above.*
- (3) *It is compatible with the forgetful map of the boundary marked points.*
- (4) *It is compatible with the permutation of interior marked points.*
- (5) *It is compatible with cyclic permutation of the boundary marked points.*
- (6) *ev_0^∂ restricted to the zero set of this system of multisections is a submersion.*

Proof. Existence of such a system of families of multisections is established in [Fu3] Corollary 5.2 by an induction over $\beta \cap \omega$ and ℓ . \square

Remark 17.5. Strictly speaking, we need to fix E_0 and ℓ_0 and restrict ourselves to those moduli spaces $\mathcal{M}_{k+1;\ell}(L; \beta)$ such that $\beta \cap \omega \leq E_0$ and $\ell \leq \ell_0$, in order to take care of the problem of ‘running out’ pointed out in [FOOO1] Subsection 7.2.3. We can handle this in the same way as [FOOO1]. In our de Rham version the way to resolving this problem is simpler than the singular homology version of [FOOO1] and is written in detail in [Fu3] Section 14 in the case $\ell = 0$. The case $\ell \neq 0$ can be handled in the same way by using the homological algebra developed in [FOOO1] Section 7.4.

Let $g_1, \dots, g_\ell \in \Omega(M)$ and $h_1, \dots, h_k \in \Omega(L)$ and β with $(\beta, \ell) \neq (0, 0)$. We define

$$\begin{aligned} & \mathfrak{q}_{\ell,k;\beta}(g_1, \dots, g_\ell, h_1, \dots, h_k) \\ &= \text{ev}_{0!}^\partial (\text{ev}_1^* g_1 \wedge \dots \wedge \text{ev}_\ell^* g_\ell \wedge \text{ev}_1^{\partial*} h_1 \wedge \dots \wedge \text{ev}_k^{\partial*} h_k). \end{aligned} \quad (17.9)$$

Here we use the evaluation map

$$(\text{ev}, \text{ev}^\partial) = (\text{ev}_1, \dots, \text{ev}_\ell; \text{ev}_0^\partial, \dots, \text{ev}_k^\partial) : \mathcal{M}_{k+1;\ell}(L; \beta) \rightarrow M^\ell \times L^{k+1}$$

and the correspondence by this moduli space in (17.9). For $\beta = \beta_0 = 0$, $\ell = 0$ we put

$$\mathfrak{q}_{0,k;\beta_0}(h_1, \dots, h_k) = \begin{cases} 0 & k \neq 1, 2 \\ (-1)^{n+1+\deg h_1} dh_1 & k = 1 \\ (-1)^{\deg h_1(\deg h_1+1)} h_1 \wedge h_2 & k = 2. \end{cases} \quad (17.10)$$

Theorem 17.1 (1) is a consequence of Proposition 17.3 (3) and the compatibility of the family of multisections with this boundary identification.

We may regard (17.4) as the definition of its right hand side. So Theorem 17.1 (2) is obvious.

Theorem 17.1 (3) is a consequence of Proposition 17.3 (8) and the compatibility of the family of multisections to this forgetful map. See [Fu3] Section 7 for the detail of this point. The proof of Theorem 17.1 is complete. \square

Remark 17.6. (1) For $g_1 \otimes \dots \otimes g_\ell \in B_\ell(\Omega(M))$ and $h_1 \otimes \dots \otimes h_k \in B_k(\Omega(L))$ we defined $\mathfrak{q}_{\ell,k;\beta}(g_1, \dots, g_\ell, h_1, \dots, h_k)$ by (17.9). Thanks to Proposition 17.3 (9) this is invariant under the permutation of g_1, \dots, g_ℓ . Thus the operator $\mathfrak{q}_{\ell,k;\beta}$ descends to the operator

$$\mathfrak{q}_{\ell,k;\beta} : E_\ell(\Omega(M)[2]) \otimes B_k(\Omega(L)[1]) \rightarrow \Omega(L)[1].$$

- (2) The coefficient on the right hand side of (17.9) is the same as in Definition 6.10 of [FOOO6], but different from one in (3.8.68) of [FOOO1] and (6.10) of [FOOO3]. In [FOOO1], [FOOO2] and [FOOO3], as we noted in Notations and Conventions (20), we denoted by $E_\ell C$ the \mathfrak{S}_ℓ -invariant subset of BC and used the deconcatenation coproduct on it. Indeed, if we denote the operator defined by (6.10) of [FOOO3] (or (3.8.68) of [FOOO1]) by $\mathfrak{q}_{\ell,k;\beta}^{\text{book}}$, we have

$$\mathfrak{q}_{\ell,k;\beta}^{\text{book}} = \frac{1}{\ell!} \mathfrak{q}_{\ell,k;\beta}.$$

However, we can see that this difference does not cause any trouble in the proof of Theorem 17.1 by just noticing the identity

$$\mathbf{y}_c^{2;1} \otimes \mathbf{y}_c^{2;2} = \frac{\ell_1! \ell_2!}{\ell!} \mathbf{y}_c^{2;1'} \otimes \mathbf{y}_c^{2;2'}$$

on $E_{\ell_1}C \otimes E_{\ell_2}C$, where the left (resp. right) hand side is the (ℓ_1, ℓ_2) -component in the decomposition of $\Delta_{\text{decon}}\mathbf{y}$ for the invariant set (resp. $\Delta_{\text{shuff}}\mathbf{y}$ for the quotient space). Here we identify the quotient set with the invariant subset by the map

$$[y_1 \otimes \cdots \otimes y_\ell] \mapsto \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^* y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(\ell)}$$

$$\text{with } * = \sum_{i < j; \sigma(i) > \sigma(j)} \deg y_i \deg y_j.$$

We next explain how we use the map \mathfrak{q} to deform the filtered A_∞ structure \mathfrak{m} on L . In this section we use the universal Novikov ring Λ_0 .

Definition 17.7. (1) Let $\mathfrak{b}_0 \in H^0(M; \Lambda_0)$, $\mathfrak{b}_{2;1} \in H^2(M, L; \mathbb{C})$, $\mathfrak{b}_+ \in H^2(M; \Lambda_+) \oplus \bigoplus_{k \geq 2} H^{2k}(M; \Lambda_0)$, $b_+ \in \Omega^1(L) \hat{\otimes} \Lambda_+ \oplus \bigoplus_{k \geq 2} \Omega^{2k-1}(L) \hat{\otimes} \Lambda_0$. We represent \mathfrak{b}_0 , \mathfrak{b}_+ by closed differential forms which are denoted by the same letters. Put $\mathbf{b} = (\mathfrak{b}_0, \mathfrak{b}_{2;1}, \mathfrak{b}_+, b_+)$.

(2) For each $k \neq 0$, we define $\mathfrak{m}_k^{\mathbf{b}}$ by

$$\begin{aligned} & \mathfrak{m}_k^{\mathbf{b}}(x_1, \dots, x_k) \\ &= \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell=0}^{\infty} \sum_{m_0=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathfrak{b}_{2;1} \cap \beta)}{\ell!} \\ & \quad \mathfrak{q}_{\ell, k + \sum_{i=0}^k m_i; \beta}(\mathfrak{b}_+^{\otimes \ell}; b_+^{\otimes m_0}, x_1, b_+^{\otimes m_1}, \dots, b_+^{\otimes m_{k-1}}, x_k, b_+^{\otimes m_k}), \end{aligned} \tag{17.11}$$

where $x_i \in \Omega(L)$. We extend it Λ_0 -linearly to $\Omega(L) \hat{\otimes} \Lambda_0$.

For $k = 0$, we define $\mathfrak{m}_0^{\mathbf{b}}$ by

$$\mathfrak{m}_0^{\mathbf{b}}(1) = \mathfrak{b}_0 + \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathfrak{b}_{2;1} \cap \beta)}{\ell!} \mathfrak{q}_{\ell, k+m; \beta}(\mathfrak{b}_+^{\otimes \ell}; b_+^{\otimes m}). \tag{17.12}$$

Here we embed $H^0(M; \Lambda_0) = \Lambda_0 \subset \Omega^0(L) \hat{\otimes} \Lambda_0$ as Λ_0 -valued constant functions on M .

We can prove that the right hand side converges in T -adic topology in the same way as in Lemma 6.5.

Lemma 17.8. *The family $\{\mathfrak{m}_k^{\mathbf{b}}\}_{k=0}^{\infty}$ defines a filtered A_∞ structure on $\Omega(L) \hat{\otimes} \Lambda_0$.*

Proof. The proof is a straightforward calculation using Theorem 17.1. See Lemma 3.8.39 [FOOO1] for the detail of the proof of such a statement in the purely abstract context. \square

We regard the constant function 1 on L as a differential 0 form and write it \mathbf{e}_L .

Definition 17.9. Denote by $\widehat{\mathcal{M}}_{\text{weak, def}}(L; \Lambda_0)$ the set of all the elements $\mathbf{b} = (\mathfrak{b}_0, \mathfrak{b}_{2;1}, \mathfrak{b}_+, b_+)$ as in Definition 17.7 such that

$$\mathfrak{m}_0^{\mathbf{b}}(1) = c \mathbf{e}_L$$

for $c \in \Lambda_+$. We define $\mathfrak{P}\mathfrak{D}(\mathbf{b}) \in \Lambda_+$ by the equation

$$\mathfrak{m}_0^{\mathbf{b}}(1) = \mathfrak{P}\mathfrak{D}(\mathbf{b}) \mathbf{e}_L.$$

We call the map $\mathfrak{P}\mathfrak{D} : \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_0) \rightarrow \Lambda_+$ the *potential function*. We also define the projection

$$\pi : \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_0) \rightarrow H^0(M; \Lambda_0) \oplus H^2(M, L; \mathbb{C}) \oplus H^2(M; \Lambda_+) \oplus \bigoplus_{k \geq 2} H^{2k}(M; \Lambda_0)$$

by

$$\pi(\mathfrak{b}_0, \mathfrak{b}_{2;1}, \mathfrak{b}_+, b_+) = (\mathfrak{b}_0, \mathfrak{b}_{2;1}, \mathfrak{b}_+).$$

Let $\mathbf{b}^{(i)} = (\mathfrak{b}_0^{(i)}, \mathfrak{b}_{2;1}^{(i)}, \mathfrak{b}_+^{(i)}, b_+^{(i)}) \in \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,\text{nov}}^+)$ ($i = 1, 2$) such that

$$\pi(\mathbf{b}^{(1)}) = \pi(\mathbf{b}^{(0)}).$$

We define an operator

$$\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}} : \Omega(L) \widehat{\otimes} \Lambda_0 \rightarrow \Omega(L) \widehat{\otimes} \Lambda_0$$

of degree +1 by

$$\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}}(x) = \sum_{k_1, k_0} \mathfrak{m}_{k_1+k_0+1}^{\overline{\mathbf{b}}} ((b_+^{(1)})^{\otimes k_1} \otimes x \otimes (b_+^{(0)})^{\otimes k_0}),$$

where $\overline{\mathbf{b}} = (\mathfrak{b}_0^{(0)}, \mathfrak{b}_{2;1}^{(0)}, \mathfrak{b}_+^{(0)}, 0) = (\mathfrak{b}_0^{(1)}, \mathfrak{b}_{2;1}^{(1)}, \mathfrak{b}_+^{(1)}, 0)$. We remark that if $\mathbf{b}_1 = \mathbf{b}_0 = \mathbf{b}$ we have

$$\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}} = \mathfrak{m}_1^{\mathbf{b}}. \quad (17.13)$$

Lemma 17.10.

$$(\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}} \circ \delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}})(x) = (-\mathfrak{P}\mathfrak{D}(\mathbf{b}^{(1)}) + \mathfrak{P}\mathfrak{D}(\mathbf{b}^{(0)}))x.$$

Proof. This is an easy consequence of Theorem 17.1. See [FOOO1] Proposition 3.7.17. \square

This enables us to give the following definition

Definition 17.11. ([FOOO1] Definition 3.8.61.) For a given pair $\mathbf{b}^{(1)}, \mathbf{b}^{(0)} \in \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_0)$ satisfying

$$\pi(\mathbf{b}^{(1)}) = \pi(\mathbf{b}^{(0)}), \quad \mathfrak{P}\mathfrak{D}(\mathbf{b}^{(1)}) = \mathfrak{P}\mathfrak{D}(\mathbf{b}^{(0)}),$$

we define

$$HF((L, \mathbf{b}^{(1)}), (L, \mathbf{b}^{(0)}); \Lambda_0) = \frac{\text{Ker}(\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}})}{\text{Im}(\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}})}.$$

When $\mathbf{b}^{(1)} = \mathbf{b}^{(0)} = \mathbf{b}$, we just write $HF((L, \mathbf{b}); \Lambda_0)$ for simplicity.

Put $CF_{\text{dR}}(L; \Lambda) = \Omega(L) \widehat{\otimes} \Lambda$. Then $(CF_{\text{dR}}(L; \Lambda), \delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}})$ forms a cochain complex. The cochain complex $CF_{\text{dR}}(L; \Lambda)$ carries a natural filtration given by

$$F^\lambda CF_{\text{dR}}(L; \Lambda) = T^\lambda \Omega(L) \widehat{\otimes} \Lambda_0. \quad (17.14)$$

Lemma 17.12. *We have*

$$\delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}}(F^\lambda CF_{\text{dR}}(L; \Lambda)) \subset F^\lambda CF_{\text{dR}}(L; \Lambda).$$

Proof. Since the symplectic area of a pseudo-holomorphic map is nonnegative, $\beta \cap \omega \geq 0$ if $\mathcal{M}_{k+1; \ell}(L; \beta)$ is nonempty. Therefore if $\mathfrak{q}_{\ell, k; \beta}$ is nonzero then $\beta \cap \omega$ is nonnegative. The lemma follows from this fact and the definition. \square

This enables us to define the following Lagrangian version of spectral numbers associated to L .

Definition 17.13. For $x \in HF((L, \mathbf{b}^{(1)}), (L, \mathbf{b}^{(0)}); \Lambda)$ we put

$$\rho_L^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}}(x) = -\sup\{\lambda \mid \exists \widehat{x} \in F^\lambda CF_{\text{dR}}(L; \Lambda), \delta^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}}(\widehat{x}) = 0, [\widehat{x}] = x \in HF((L, \mathbf{b}^{(1)}), (L, \mathbf{b}^{(0)}); \Lambda)\}. \quad (17.15)$$

Remark 17.14. We put minus sign in (17.15) for the sake of consistency with Chapters 2 and 3. In fact, $\mathbf{v}_q = -\mathbf{v}_T$ via the isomorphism $\Lambda^\downarrow \cong \Lambda$.

We can show

$$\rho_L^{\mathbf{b}^{(1)}, \mathbf{b}^{(0)}}(x) \neq -\infty \quad (17.16)$$

if $x \neq 0$. (See [Us1] or Lemma 18.17 of this paper for the detail.)

We next define an open-closed map from the cohomology of the ambient space to the Floer cohomology of L . Let $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+) \in \widehat{\mathcal{M}}_{\text{weak, def}}(L; \Lambda_0)$, take $g \in \Omega(M)$ and define a map $i_{\text{qm}, \mathbf{b}}(g) : \Omega(M) \otimes \Lambda_0 \rightarrow CF_{\text{dR}}(L; \Lambda_0)$ by

$$i_{\text{qm}, \mathbf{b}}(g) = (-1)^{\deg g} \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sum_{k=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{(\ell_1 + \ell_2 + 1)!} \mathfrak{q}_{\ell, k + \sum_{i=0}^k m_i; \beta}(\mathbf{b}_+^{\otimes \ell_1} g \mathbf{b}_+^{\otimes \ell_2}; b_+^{\otimes k}). \quad (17.17)$$

It follows in the same way as in Lemma 6.5 that the right hand side converges in T -adic topology.

Lemma 17.15. *The map $i_{\text{qm}, \mathbf{b}}$ is a chain map. Namely,*

$$\delta^{\mathbf{b}, \mathbf{b}} \circ i_{\text{qm}, \mathbf{b}} = \pm i_{\text{qm}, \mathbf{b}} \circ d.$$

Proof. This is a consequence of Theorem 17.1. See [FOOO1] Theorem 3.8.62. We recall from Remark 3.5.8 of [FOOO1] that $\mathbf{m}_{1; \beta_0}$ in (17.4) satisfies $\mathbf{m}_{1; \beta_0}(h) = (-1)^{n + \deg h + 1} dh$ for $h \in \Omega^{\deg h}(M)$. \square

We thus obtain a homomorphism

$$i_{\text{qm}, \mathbf{b}}^* : H^*(M; \Lambda_0) \rightarrow HF^*((L, \mathbf{b}); \Lambda_0). \quad (17.18)$$

Remark 17.16. The homomorphism (17.18) is indeed a ring homomorphism. It is proved in [FOOO6] Section 9 for the toric case. See [FOOO6] Section 31 and [AFOOO] for the general case.

Combining the map $i_{\text{qm}, \mathbf{b}}^*$ and Definition 17.13, we introduce

Definition 17.17. For each $0 \neq a \in H^*(M; \Lambda)$, we define

$$\rho_L^{\mathbf{b}}(a) = \rho_L^{\mathbf{b}, \mathbf{b}}(i_{\text{qm}, \mathbf{b}}^*(a)) \quad (17.19)$$

for $\mathbf{b} \in \widehat{\mathcal{M}}_{\text{weak, def}}(L; \Lambda_0)$.

Therefore by the finiteness (17.16), $\rho_L^{\mathbf{b}}(a) > -\infty$ for any $a \neq 0$, provided there exists some \mathbf{b} such that $i_{\text{qm}, \mathbf{b}}^*(a) \neq 0$.

18. CRITERION FOR HEAVINESS OF LAGRANGIAN SUBMANIFOLDS

In this section, we incorporate the Lagrangian Floer theory into the theory of spectral invariants and Calabi quasimorphisms of Hamiltonian flows and symplectic quasi-states.

18.1. Statement of the results. We review the notions of heavy and superheavy subsets of a symplectic manifold (M, ω) introduced by Entov and Polterovich [EP3] Definition 1.3. (See also [Al], [BC] for some related results.)

Definition 18.1. Let ζ be a partial symplectic quasistate on (M, ω) . A closed subset $Y \subset M$ is called ζ -heavy if

$$\zeta(H) \leq \sup\{H(p) \mid p \in Y\} \quad (18.1)$$

for any $H \in C^0(M)$.

A closed subset $Y \subset M$ is called ζ -superheavy if

$$\zeta(H) \geq \inf\{H(p) \mid p \in Y\} \quad (18.2)$$

for any $H \in C^0(M)$.

- Remark 18.2.** (1) Due to the different sign conventions from [EP3] as mentioned in Remark 1.2, Remark 4.17 and also because we use quantum *cohomology* class a in the definition of the spectral invariants $\rho(H; a)$, the above definition looks opposite to that of [EP3]. However after taking these different convention and usage, this definition of heavyness or of superheavyness of a given subset $S \subset (M, \omega)$ indeed is equivalent to that of [EP3].
- (2) Following the proof of Proposition 4.1 [EP3], we can obtain a characterization of a ζ -heavy set or a ζ -superheavy set as follows: A closed subset $Y \subset M$ is ζ -heavy if and only if for every $H \in C^\infty(M)$ with $H|_Y = 0$, $H \geq 0$ one has $\zeta(H) = 0$. A closed subset $Y \subset M$ is ζ -superheavy if and only if for every $H \in C^\infty(M)$ with $H|_Y = 0$, $H \leq 0$ one has $\zeta(H) = 0$. Due to the different sign convention again, this statement is slightly different from Proposition 4.1 [EP3]. Using this characterization and our triangle inequality Definition 13.3 (8) and the monotonicity (3), we can show that every ζ -superheavy subset is ζ -heavy. This is nothing but Proposition 4.2 [EP3].
- (3) Furthermore, we can show Proposition 4.3 [EP3] as it is. Namely for any ζ -superheavy set Y , and any $\alpha \in \mathbb{R}$ and $H \in C^\infty(M)$ with $H|_Y = \alpha$ we have $\zeta(H) = \alpha$.
- (4) Entov-Polterovich Theorem 1.4 (iii) [EP3] proved that for any partial symplectic quasistate ζ , every ζ -superheavy set intersects every ζ -heavy subset. See Theorem 18.7.

The definitions of heavyness and super-heavyness [EP3] involve only time *independent* Hamiltonian. We first enhance the definition by involving time-dependent Hamiltonian. For this purpose, the following definition is useful.

Definition 18.3. Let $H : [0, 1] \times M \rightarrow \mathbb{R}$ be a Hamiltonian and $Y \subset M$ be a closed subset. For such a pair (H, Y) we associate two constants $E_\infty^\pm(H; Y)$ by

$$\begin{aligned} E_\infty^-(H; Y) &= -\inf\{H(t, p) \mid (t, p) \in [0, 1] \times Y\} \\ &= \sup\{-H(t, p) \mid (t, p) \in [0, 1] \times Y\} \\ E_\infty^+(H; Y) &= \sup\{H(t, p) \mid (t, p) \in [0, 1] \times Y\} \\ E_\infty(H; Y) &= E_\infty^-(H; Y) + E_\infty^+(H; Y). \end{aligned} \quad (18.3)$$

Here the subscript ‘ ∞ ’ stands for the L^∞ -norm and used against the more natural $L^{(1,\infty)}$ -norm which we have used before and denoted as E^\pm . We note

$$E_\infty^\pm(\underline{H}; Y) = E_\infty^\pm(H; Y) \mathfrak{p} \frac{1}{\text{vol}_\omega(M)} \text{Cal}(H)$$

and so $E_\infty^-(H; Y) + E_\infty^+(H; Y) = E_\infty^-(\underline{H}; Y) + E_\infty^+(\underline{H}; Y)$ depend only on the Hamiltonian path ϕ_H , but not on the normalization constant. Therefore we denote

$$E_\infty(\phi_H; Y) = E_\infty^-(H; Y) + E_\infty^+(H; Y).$$

Definition 18.4. For $\tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$, we define

$$\begin{aligned} e_\infty^-(\tilde{\psi}; Y) &= \inf_H \{E_\infty^-(\underline{H}; Y) \mid \tilde{\psi} = [\phi_H]\} \\ e_\infty^+(\tilde{\psi}; Y) &= \inf_H \{E_\infty^+(\underline{H}; Y) \mid \tilde{\psi} = [\phi_H]\} \\ e_\infty(\tilde{\psi}; Y) &= \inf_H \{E_\infty(H; Y) \mid \tilde{\psi} = [\phi_H]\} \end{aligned} \quad (18.4)$$

We note $e_\infty(\tilde{\psi}; Y) \geq e_\infty^+(\tilde{\psi}; Y) + e_\infty^-(\tilde{\psi}; Y)$.

Definition 18.5. Let $\mu : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$ be an Entov-Polterovich pre-quasismorphism. A closed subset $Y \subset M$ is called μ -heavy if we have

$$\mu(\tilde{\psi}) \geq -\text{vol}_\omega(M) e_\infty^+(\tilde{\psi}; Y) \quad (18.5)$$

for any $\tilde{\psi}$.

A closed subset $Y \subset M$ is called μ -superheavy if we have

$$\mu(\tilde{\psi}) \leq \text{vol}_\omega(M) e_\infty^-(\tilde{\psi}; Y) \quad (18.6)$$

for any $\tilde{\psi}$.

Remark 18.6. We consider $\tilde{\psi} = [\phi_H]$ for an autonomous H . Suppose μ and ζ are related as in (14.5), then

$$-e_\infty^+(\tilde{\psi}; Y) \geq -E_\infty^+(\underline{H}; Y) = -E_\infty^+(H; Y) + \frac{1}{\text{vol}_\omega(M)} \text{Cal}(H)$$

and

$$\mu(\tilde{\psi}) = \mu(\phi_H) = -\text{vol}_\omega(M) \zeta(\underline{H}) = -(\text{vol}_\omega(M) \zeta(H) - \text{Cal}(H))$$

for autonomous H . Therefore μ -heaviness of L implies ζ -heaviness of L . Similarly, we can also see that μ -superheaviness implies ζ -superheaviness.

The following result is due to Entov-Polterovich [EP3] which will be used later in Section 23. We give a proof for reader's convenience.

Theorem 18.7 ([EP3] Theorem 1.4). *Let ζ be a partial symplectic quasistate. If $Y \subset M$ is ζ -superheavy and $Z \subset M$ is ζ -heavy, then for any $\psi \in \text{Symp}_0(M, \omega)$ we have*

$$\psi(Y) \cap Z \neq \emptyset.$$

Proof. Since superheaviness is invariant under symplectic diffeomorphisms contained in $\text{Symp}_0(M, \omega)$, we may assume that ψ is identity. Suppose $Y \cap Z = \emptyset$. We define $H : M \rightarrow \mathbb{R}$ such that $H = 1$ on Y and $H = -1$ on Z . Then since Y is ζ -superheavy, $\zeta(H) \geq \inf\{H(y) \mid y \in Y\} = 1$. On the other hand, since Z is ζ -heavy, we have $\zeta(H) \leq \sup\{H(z) \mid z \in Z\} = -1$. This is a contradiction. \square

Now the following is the main theorem of this paper whose proof is completed in Subsection 18.5.

Theorem 18.8. *Let $L \subset M$ be a relatively spin compact Lagrangian submanifold, and $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+) \in \widehat{\mathcal{M}}_{\text{weak, def}}(L; \Lambda_0)$ as in Definition 17.9. We put*

$$\mathbf{b} = i^*(\mathbf{b}_{2;1}) + \mathbf{b}_+ \in H^{\text{even}}(M; \Lambda_0),$$

where $i^* : H^2(M, L; \Lambda_0) \rightarrow H^2(M; \Lambda_0)$. Let $e \in H(M; \Lambda)$.

(1) *If $e \cup^{\mathbf{b}} e = e$ and*

$$i_{\text{qm}, \mathbf{b}}^*(e) \neq 0 \in HF^*((L, \mathbf{b}); \Lambda), \quad (18.7)$$

then L is $\zeta_e^{\mathbf{b}}$ -heavy and is $\mu_e^{\mathbf{b}}$ -heavy.

(2) *If there is a direct factor decomposition $QH_{\mathbf{b}}^*(M; \Lambda) \cong \Lambda \times Q'$ as a ring and e comes from a unit of the direct factor Λ which satisfies (18.7), then L is $\zeta_e^{\mathbf{b}}$ -superheavy and is $\mu_e^{\mathbf{b}}$ -superheavy.*

18.2. Floer homologies of periodic Hamiltonians and of Lagrangian submanifolds. The main part of the proof of Theorem 18.8 is the proof of the following proposition.

Proposition 18.9. *Let L , \mathbf{b} , and \mathbf{b} be as in Theorem 18.8 and $a \in H(M; \Lambda)$. Then*

$$\rho^{\mathbf{b}}(H; a) \geq -E_{\infty}^+(H; L) + \rho_L^{\mathbf{b}}(a) \quad (18.8)$$

for any Hamiltonian H . Here $\rho_L^{\mathbf{b}}(a)$ is as in (17.19). Equivalently, we have

$$\rho^{\mathbf{b}}(\tilde{\psi}; a) \geq -e_{\infty}^+(\tilde{\psi}; L) + \rho_L^{\mathbf{b}}(a). \quad (18.9)$$

To prove Proposition 18.9 we introduce a map $\mathfrak{J}_{(F, J)}^{\mathbf{b}, \mathbf{b}} : CF(M, H; \Lambda^{\downarrow}) \rightarrow CF_{\text{dR}}(L; \Lambda^{\downarrow})$ in Definition 18.16 and explore its properties Propositions 18.19 and 18.21. The proof of Proposition 18.9 is completed in Subsection 18.5. To define the map $\mathfrak{J}_{(F, J)}^{\mathbf{b}, \mathbf{b}}$ we start with introducing some moduli spaces.

We put

$$R = \sup\{H(t, p) \mid (t, p) \in [0, 1] \times L\}. \quad (18.10)$$

Let $\epsilon > 0$ and U a sufficiently small open neighborhood of L .

Lemma 18.10. *Let H , L and R be as above. Then for any given $\epsilon > 0$ there exists an open neighborhood U of L and a smooth function $F = F(\tau, t, x) : (-\infty, 0] \times [0, 1] \times M \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} F(\tau, t, x) &= H(t, x), & \text{if } \tau < -10, \\ F(\tau, t, x) &= R + \epsilon, & \text{if } \tau > -1 \text{ and } x \in U, \\ \frac{\partial F}{\partial \tau}(\tau, t, x) &\geq 0, & \text{for any } \tau, x. \end{aligned}$$

Proof. Using the continuity of the function $H(t, x)$ and compactness of Y and the definition of R we can choose open neighborhood U , $V \supset L$ with $\bar{U} \subset V$ so small that

$$H(t, x) < R + \frac{\epsilon}{2} \quad (18.11)$$

for $(t, x) \in [0, 1] \times V$ and fix a cut-off function $\rho_{U, V} : X \rightarrow \mathbb{R}$ such that $\rho_{U, V} = 1$ on \bar{U} and $\rho_{U, V} \equiv 0$ outside V .

We first consider the function $F_V : (-\infty, 0] \times [0, 1] \times V \rightarrow \mathbb{R}$ by

$$F_V(\tau, t, x) = \chi(\tau + 5)(R + \varepsilon) + (1 - \chi(\tau + 5))H(t, x)$$

and then define F by

$$F(\tau, t, x) = \rho_{U,V}(x)F_V(\tau, t, x) + (1 - \rho_{U,V}(x))H(t, x),$$

where χ is as in Definition 3.11.

The first two conditions required on F are obvious. It remains to check the third. We compute

$$\frac{\partial F}{\partial \tau}(\tau, t, x) = \rho_{U,V}(x) \frac{\partial F_V}{\partial \tau}(\tau, t, x) = \chi'(\tau + 5) \rho_{U,V}(x) ((R + \varepsilon) - H(t, x)).$$

If $x \in X \setminus V$, $\rho_{U,V}(x) \equiv 0$. On the other hand, if $x \in V$, we have

$$\rho_{U,V}(x)(R + \varepsilon - H(t, x)) \geq \rho_{U,V}(x) \times \frac{\varepsilon}{2} \geq 0$$

by (18.11). This finishes the proof. \square

We recall that we fix a t -independent J throughout in Chapter 4.

Definition 18.11. Let $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$ and $\beta \in H_2(M, L; \mathbb{Z})$. We denote by $\mathring{\mathcal{M}}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ the set of all triples $(u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k)$ satisfying the following:

- (1) $u : (-\infty, 0] \times S^1 \rightarrow M$ is a smooth map such that $u(0, t) \subset L$.
- (2) The map u satisfies the equation

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_F(u) \right) = 0. \quad (18.12)$$

- (3) The energy

$$E_{(F,J);L} = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_J^2 + \left| \frac{\partial u}{\partial t} - X_F(u) \right|_J^2 \right) dt d\tau$$

is finite.

- (4) The map u satisfies the following asymptotic boundary condition.

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = \gamma(t). \quad (18.13)$$

- (5) z_1^+, \dots, z_ℓ^+ are points in $(-\infty, 0) \times S^1$ which are mutually distinct.
- (6) z_0, \dots, z_k are points on the boundary $\{0\} \times S^1$. They are mutually distinct. z_0, \dots, z_k respects the counterclockwise cyclic order on S^1 with respect to the boundary orientation on S^1 coming from $(-\infty, 0] \times S^1$. We always set $z_0 = (0, 0)$.
- (7) The homology class of the concatenation of w and u is β .

We define an evaluation map

$$(\text{ev}, \text{ev}^\partial) = (\text{ev}_1, \dots, \text{ev}_\ell; \text{ev}_0^\partial, \dots, \text{ev}_k^\partial) : \mathring{\mathcal{M}}_{k+1;\ell}(F, J; [\gamma, w], L; \beta) \rightarrow M^\ell \times L^{k+1}$$

where

$$\text{ev}_i([u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k]) = u(z_i^+), \quad \text{ev}_i^\partial([u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k]) = u(z_i).$$

Lemma 18.12. (1) *The moduli space $\mathring{\mathcal{M}}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ has a compactification $\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ that is Hausdorff.*

- (2) The space $\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ has an orientable Kuranishi structure with corners.
- (3) The boundary of $\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ is described by the union of the following two types of fiber or direct products.

$$\bigcup \mathcal{M}_{\#\mathbb{L}_1}(H, J; [\gamma, w], [\gamma', w']) \times \mathcal{M}_{k+1;\#\mathbb{L}_2}(F, J; [\gamma', w'], L; \beta), \quad (18.14)$$

where the union is taken over all $(\gamma', w') \in \text{Crit}(\mathcal{A}_H)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\bigcup \mathcal{M}_{k_1+1;\#\mathbb{L}_1}(L; \beta_1)_{\text{ev}_0^\partial} \times_{\text{ev}_i^\partial} \mathcal{M}_{k_2+1;\#\mathbb{L}_2}(F, J; [\gamma, w], L; \beta_2), \quad (18.15)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, k_1, k_2 with $k_1 + k_2 = k$, $i \leq k_2$, and β_1, β_2 with $\beta_1 + \beta_2 = \beta$.

- (4) Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ be the Conley-Zehnder index and $\mu_L : H_2(M, L; \mathbb{Z}) \rightarrow 2\mathbb{Z}$ the Maslov index. Then the (virtual) dimension satisfies the following equality:

$$\dim \mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta) = \mu_L(\beta) - \mu_H([\gamma, w]) + 2\ell + k - 2 + n. \quad (18.16)$$

- (5) We can define orientations of $\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ so that (3) above is compatible with this orientation. The compatibility for the boundary of type (18.15) is in the sense of [FOOO1] Proposition 8.3.3.
- (6) The evaluation map $(\text{ev}, \text{ev}^\partial)$ extends to a strongly continuous smooth map

$$\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta) \rightarrow M^\ell \times L^{k+1},$$

which we denote also by the same symbol. It is compatible with (3).

- (7) ev_0^∂ is weakly submersive.
- (8) The Kuranishi structure is compatible with forgetful map of the boundary marked points.
- (9) The Kuranishi structure is invariant under the permutation of the interior marked points.
- (10) The Kuranishi structure is invariant under the cyclic permutation of the boundary marked points.

The proof of Lemma 18.12 is the same as those of Propositions 3.6, 17.3 and so is omitted.

Remark 18.13. A similar moduli space was used by Albers [Al] in the monotone case. According to Entov-Polterovich [EP3] p.779, their motivation to define heavyness comes from [Al]. We note that Albers [Al] does not use τ -dependent $F = F(\tau, t, x)$ but $H = H(t, x)$ which is independent of τ . We may not need F and simply use X_H -perturbed pseudo-holomorphic curve equation in place of (18.12). (See Section 25.4 for more explanation on this point.) The reason why we use this F is to deal with genuine pseudo-holomorphic curve equation in a neighborhood of the boundary point $\{0\} \times S^1$ where the boundary condition $u(0, t) \in L$ is put: since F is constant there, this does our purpose. The detail of the compactification and gluing in the study of moduli space of X_H perturbed pseudo-holomorphic curve equation and Lagrangian boundary condition does not seem to be written in detail in the literature in the level of generality we need here, although we have no doubt that there is nothing novel to work out. Since (18.12) is exactly the pseudo-holomorphic curve equation in a neighborhood of the boundary point $\{0\} \times S^1$,

we can directly apply the analysis of [FOOO1] Section 7.2 for the proof of Lemma 18.12.

Lemma 18.14. *There exists a system of continuous families of multisections on our moduli space $\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ with the following properties.*

- (1) *It is transversal to 0.*
- (2) *It is compatible with the description of the boundary in Proposition 18.12*
- (3).
- (3) *The restriction of ev_0^∂ to the zero set of this family of multisections is a submersion.*
- (4) *It is compatible with forgetful map of the boundary marked points.*
- (5) *It is invariant under the permutation of the interior marked points.*
- (6) *It is invariant under the cyclic permutation of the boundary marked points.*

The proof of Lemma 18.14 is similar to the proof of Lemma 17.4 and so omitted.

Let $CF(M, H; \mathbb{C})$ be the \mathbb{C} vector space over the basis $\text{Crit}(\mathcal{A}_H)$. We use our moduli space to define an operator

$$\mathbf{q}_{\ell,k;\beta}^F : E_\ell(\Omega(M)[2]) \otimes CF(M, H; \mathbb{C})[1] \otimes B_k(\Omega(L)[1]) \rightarrow \Omega(L)[1] \quad (18.17)$$

as follows. Let $g_1, \dots, g_\ell \in \Omega(M)$, $h_1, \dots, h_k \in \Omega(L)$, $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$, and $\beta \in H_2(M, L; \mathbb{Z})$. We define

$$\begin{aligned} & \mathbf{q}_{\ell,k;\beta}^F(g_1, \dots, g_\ell, [\gamma, w], h_1, \dots, h_k) \\ &= \text{ev}_{0!}^\partial (\text{ev}_1^* g_1 \wedge \dots \wedge \text{ev}_\ell^* g_\ell \wedge \text{ev}_1^{\partial*} h_1 \wedge \dots \wedge \text{ev}_k^{\partial*} h_k). \end{aligned} \quad (18.18)$$

Here we use the evaluation map

$$(\text{ev}, \text{ev}^\partial) = (\text{ev}_1, \dots, \text{ev}_\ell; \text{ev}_0^\partial, \dots, \text{ev}_k^\partial) : \mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta) \rightarrow M^\ell \times L^{k+1}$$

and the correspondence given by this moduli space. The next proposition states the main property of this operator. We define

$$\partial_{(H,J),\beta} : E(\Omega(M)[2]) \otimes CF(M, H; \mathbb{C})[1] \rightarrow CF(M, H; \mathbb{C})[1]$$

by

$$\begin{aligned} & \partial_{(H,J),\beta}(g_1, \dots, g_\ell; [\gamma, w]) \\ &= \sum_{[\gamma', w']} \mathbf{n}_{(F,J);\ell}([\gamma, w], [\gamma', w'])(g_1, \dots, g_\ell)[\gamma', w'], \end{aligned} \quad (18.19)$$

where $\mathbf{n}_{(F,J);\ell}([\gamma, w], [\gamma', w'])(g_1, \dots, g_\ell)$ is (6.4).

Proposition 18.15. *The operators $\mathbf{q}_{\ell,k;\beta}^F$ have the following properties:*

- (1) $\mathbf{q}_{\ell,k;\beta}^F$ satisfies

$$\begin{aligned} 0 &= \sum_{\beta_1 + \beta_2 = \beta} \sum_{c_1, c_2} (-1)^* \mathbf{q}_{\beta_1}(\mathbf{y}_{c_1}^{2;1}; \mathbf{x}_{c_2}^{3;1} \otimes \mathbf{q}_{\beta_2}^F(\mathbf{y}_{c_1}^{2;2}, [\gamma, w]; \mathbf{x}_{c_2}^{3;2}) \otimes \mathbf{x}_{c_2}^{3;3}) \\ &+ \sum_{\beta_1 + \beta_2 = \beta} \sum_{c_1, c_2} (-1)^{**} \mathbf{q}_{\beta_1}^F(\mathbf{y}^{2;1}, \partial_{(H,J),\beta_2}(\mathbf{y}^{2;2}, [\gamma, w]); \mathbf{x}) \end{aligned} \quad (18.20)$$

where

$$* = \deg' \mathbf{x}_{c_2}^{3;1} + \deg' \mathbf{x}_{c_2}^{3;1} \deg \mathbf{y}_{c_1}^{2;2} + \deg \mathbf{y}_{c_1}^{2;1}, \quad ** = \deg \mathbf{y}_{c_1}^{2;1}.$$

In (18.20) and hereafter, we simplify our notation by writing $\mathbf{q}_\beta^F(\mathbf{y}; [\gamma, w]; \mathbf{x})$, $\mathbf{q}_\beta(\mathbf{y}; \mathbf{x})$ in place of $\mathbf{q}_{\ell,k;\beta}^F(\mathbf{y}; [\gamma, w]; \mathbf{x})$, $\mathbf{q}_{\ell,k;\beta}(\mathbf{y}; \mathbf{x})$ if $\mathbf{y} \in E_\ell(\Omega(M)[2])$, $\mathbf{x} \in B_k(\Omega(L)[1])$. We use the notation (17.2) here.

- (2) Let \mathbf{e}_L be the constant function 1 which we regard degree 0 differential form on L . Let $\mathbf{x}_i \in B(H(L; R)[1])$ and we put $\mathbf{x} = \mathbf{x}_1 \otimes \mathbf{e}_L \otimes \mathbf{x}_2 \in B(H(L; R)[1])$. Then

$$\mathfrak{q}_\beta^F(\mathbf{y}, [\gamma, w]; \mathbf{x}) = 0. \quad (18.21)$$

Proof. (1) follows from Lemma 18.14 (2) and Proposition 18.12 (3). (2) follows from 18.14 (4) and Proposition 18.12 (8). \square

Let $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+)$ as in Definition 17.7. We put $\mathbf{b} = i^*(\mathbf{b}_{2;1}) + \mathbf{b}_+$. Hereafter in this section we identify the universal Novikov ring Λ with Λ^\downarrow by $T = q^{-1}$ and use the later.

Definition 18.16. We define

$$\mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}} : CF(M, H; \Lambda^\downarrow) \rightarrow CF_{\text{dR}}(L; \Lambda^\downarrow)$$

by

$$\begin{aligned} \mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}}([\gamma, w]) &= \sum_{\beta} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} q^{-(\beta \cap \omega - w \cap \omega)} \frac{\exp(\mathbf{b}_{2;1} \cap \beta - i^*(\mathbf{b}_{2;1}) \cap w)}{\ell!} \\ &\quad \mathfrak{q}_{\ell,k;\beta}^F(\mathbf{b}_+^{\otimes \ell}, [\gamma, w], b_+^{\otimes k}). \end{aligned} \quad (18.22)$$

We can prove the convergence of the right hand side of (18.22) in q -adic topology in the same way as in Lemma 6.5.

Lemma 18.17. *We have*

$$\delta^{\mathbf{b}} \circ \mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}} = \mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}} \circ \partial_{(H,J)}^{\mathbf{b}}.$$

The proof is a straightforward calculation using Proposition 18.15 and so omitted. This gives rise to a map

$$\mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b},*} : HF_*(M, H; \Lambda^\downarrow) \rightarrow HF^*((L, \mathbf{b}); \Lambda^\downarrow). \quad (18.23)$$

Remark 18.18. We can show that the map (18.23) is a ring homomorphism with respect to the pants product in the left hand side and \mathfrak{m}_2 in the right hand side. We do not prove it here since we do not use it.

18.3. Filtration and the map $\mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}}$. Using the identification $CF_{\text{dR}}(L; \Lambda^\downarrow) = CF_{\text{dR}}(L; \Lambda)$ via $T = q^{-1}$ we define a filtration on them by

$$F^{-\lambda} CF_{\text{dR}}(L; \Lambda^\downarrow) = F^\lambda CF_{\text{dR}}(L; \Lambda) = T^\lambda \Omega(L) \hat{\otimes} \Lambda_0.$$

Similarly we put

$$F^\lambda (\Omega(M) \hat{\otimes} \Lambda^\downarrow) = q^{-\lambda} \Omega(M) \hat{\otimes} \Lambda_0^\downarrow.$$

This is consistent with Definitions 2.2, 2.4. See Notations and Conventions (16) in Section 1 and also Remark 17.14. In this subsection we prove the following:

Proposition 18.19. *For all $\lambda \in \mathbb{R}$,*

$$\mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}}(F^\lambda (\Omega(M) \hat{\otimes} \Lambda^\downarrow)) \subseteq F^{\lambda+R+\epsilon} CF_{\text{dR}}(L; \Lambda^\downarrow).$$

Proof. The proposition immediately follows from Lemma 18.20 below. \square

Lemma 18.20. *If $\mathcal{M}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$ is nonempty, we have*

$$\mathcal{A}_H([\gamma, w]) \geq -(R + \epsilon) - \beta \cap \omega.$$

Proof. Let $(u; z_1^+, \dots, z_\ell^+, z_0, \dots, z_k) \in \mathring{\mathcal{M}}_{k+1;\ell}(F, J; [\gamma, w], L; \beta)$. In a way similar to the proof of Lemma 9.8, we calculate

$$\begin{aligned}
\int u^* \omega &= E_{F,J}(u) - \int_{(-\infty, 0] \times S^1} \frac{\partial}{\partial \tau} (F \circ u) d\tau dt + \int_{(-\infty, 0] \times S^1} \left(\frac{\partial F}{\partial \tau} \circ u \right) d\tau dt \\
&\geq \lim_{\tau \rightarrow -\infty} \int_{S^1} F(\tau, t, u(\tau, t)) dt \\
&\quad - \int_{S^1} F(0, t, (u(0, t))) dt + \int_{(-\infty, 0] \times S^1} \left(\frac{\partial F}{\partial \tau} \circ u \right) d\tau dt \\
&= \int_{S^1} H(t, \gamma(t)) dt - (R + \varepsilon) + \int_{(-\infty, 0] \times S^1} \left(\frac{\partial F}{\partial \tau} \circ u \right) d\tau dt \\
&\geq \int_{S^1} H_t(\gamma(t)) dt - (R + \epsilon).
\end{aligned}$$

Recalling $\beta \cap \omega - \int w^* \omega = \int u^* \omega$ from $\beta = [w \# u]$, we obtain

$$- \int w^* \omega - \int_{S^1} H_t(\gamma(t)) dt \geq -(R + \epsilon) - \beta \cap \omega.$$

The lemma follows. \square

18.4. Identity $\mathfrak{J}_{(F,J)}^{\mathbf{b}, \mathbf{b}, *}, \circ \mathcal{P}_{(H_\chi, J), *}^{\mathbf{b}} = i_{\text{qm}, \mathbf{b}}^*$. In this subsection we prove:

Proposition 18.21. *For any $a \in H^*(M; \Lambda^\downarrow)$ the identity*

$$\mathfrak{J}_{(F,J)}^{\mathbf{b}, \mathbf{b}, *} \circ \mathcal{P}_{(H_\chi, J), *}^{\mathbf{b}}(a^{\mathbf{b}}) = i_{\text{qm}, \mathbf{b}}^*(a)$$

holds in cohomology. Here $a^{\mathbf{b}} \in H_(M; \Lambda^\downarrow)$ is the homology class Poincaré dual to $a \in H^*(M; \Lambda^\downarrow)$ as in Notations and Conventions (17).*

Proof. For $S \geq 0$ we put

$$F_S(\tau, t, x) = \chi(\tau + S + 20)F(\tau, t, x)$$

where χ is as in Definition 3.11.

Definition 18.22. Denote by

$$\mathring{\mathcal{M}}_{k+1;\ell}(F_S, J; *, L; \beta)$$

the set of all triples $(u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k)$ satisfying the following:

- (1) $u : (-\infty, 0] \times S^1 \rightarrow M$ is a smooth map such that $u(0, t) \subset L$.
- (2) u satisfies the equation

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{F_S}(u) \right) = 0. \quad (18.24)$$

- (3) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_J^2 + \left| \frac{\partial u}{\partial t} - X_{F_S}(u) \right|_J^2 \right) dt d\tau$$

is finite.

- (4) z_1^+, \dots, z_ℓ^+ are points in $(-\infty, 0) \times S^1$ which are mutually distinct.
- (5) z_0, \dots, z_k are points on the boundary $\{0\} \times S^1$. They are mutually distinct. z_0, \dots, z_k respects the counterclockwise cyclic order on S^1 . We always set $z_0 = (0, 0)$.

(6) The homology class of u is β .

We define an evaluation map

$$(\text{ev}, \text{ev}^\partial) = (\text{ev}_1, \dots, \text{ev}_\ell; \text{ev}_0^\partial, \dots, \text{ev}_k^\partial) : \mathring{\mathcal{M}}_{k+1;\ell}(F_S, J; *, L; \beta) \rightarrow M^\ell \times L^{k+1}$$

by

$$\text{ev}_i(u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k) = u(z_i^+), \quad \text{ev}_i^\partial([u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k]) = u(z_i).$$

We also define

$$\text{ev}_{-\infty} : \mathring{\mathcal{M}}_{k+1;\ell}(F_S, J; *, L; \beta) \rightarrow M$$

by

$$\text{ev}_{-\infty}(u; z_1^+, \dots, z_\ell^+; z_0, \dots, z_k) = \lim_{\tau \rightarrow -\infty} u(\tau, t).$$

By (2), (3) and the removable singularity theorem, the limit of the right hand side exists and is independent of t . We put

$$\mathring{\mathcal{M}}_{k+1;\ell}(\text{para}; F, J; *, L; \beta) = \bigcup_{S \in [0, \infty)} \{S\} \times \mathring{\mathcal{M}}_{k+1;\ell}(F_S, J; *, L; \beta), \quad (18.25)$$

where $(\text{ev}, \text{ev}^\partial)$ and $\text{ev}_{-\infty}$ are defined on it in an obvious way.

- Lemma 18.23.** (1) *The moduli space $\mathring{\mathcal{M}}_{k+1;\ell}(\text{para}; F, J; *, L; \beta)$ has a compactification $\mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta)$ that is Hausdorff.*
(2) *The space $\mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta)$ has an orientable Kuranishi structure with corners.*
(3) *The boundary of $\mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta)$ is described by the union of the three types of direct or fiber products:*

$$\bigcup \mathcal{M}_{\#\mathbb{L}_1}(H_\chi, J; *, [\gamma, w]) \times \mathcal{M}_{k+1;\#\mathbb{L}_2}(F, J; [\gamma, w], L; \beta), \quad (18.26)$$

where the union is taken over all $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$, $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$. (Here $\mathcal{M}_{\#\mathbb{L}_1}(H_\chi, J; *, [\gamma, w])$ is the moduli space defined in Definition 6.10 and Proposition 6.11. We write J in place of J_χ since in Chapter 4 we use a fixed J which is independent of t and τ .)

$$\bigcup \mathcal{M}_{k_1+1;\#\mathbb{L}_1}(L; \beta_1)_{\text{ev}_0^\partial} \times_{\text{ev}_i^\partial} \mathcal{M}_{k_2+1;\#\mathbb{L}_2}(\text{para}; F, J; *, L; \beta_2), \quad (18.27)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, k_1, k_2 with $k_1 + k_2 = k$, $i \leq k_2$, and β_1, β_2 with $\beta_1 + \beta_2 = \beta$.

$$\mathcal{M}_{k+1;\ell}(F_0, J; *, L; \beta), \quad (18.28)$$

that is a compactification of the $S = 0$ case of the moduli space $\mathring{\mathcal{M}}_{k+1;\ell}(F_S, J; *, L; \beta)$.

- (4) *Let $\mu_L : H_2(M, L; \mathbb{Z}) \rightarrow 2\mathbb{Z}$ be the Maslov index. Then the (virtual) dimension satisfies the following equality:*

$$\dim \mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta) = \mu_L(\beta) + 2\ell + k - 1 + n. \quad (18.29)$$

- (5) *We can define orientations of $\mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta)$ so that (3) above is compatible with this orientation. For the boundary of type (18.27) the compatibility means the same as in Lemma 18.12 (5).*
(6) *$(\text{ev}, \text{ev}^\partial, \text{ev}_{-\infty})$ extends to a strongly continuous smooth map*

$$\mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta) \rightarrow M^{\ell+1} \times L^{k+1},$$

which we denote by the same symbol. It is compatible with (3).

- (7) ev_0^∂ is weakly submersive.
- (8) The Kuranishi structure is compatible with forgetful map of the boundary marked points.
- (9) The Kuranishi structure is invariant under the permutation of the interior marked points.
- (10) The Kuranishi structure is invariant under the cyclic permutation of the boundary marked points.

The proof of Lemma 18.23 is the same as that of Propositions 3.6. It suffices to observe that (18.26) appears at the limit $S \rightarrow \infty$.

Lemma 18.24. *There exists a system of continuous families of multisections on our moduli spaces $\mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta)$ with the following properties.*

- (1) It is transversal to 0.
- (2) It is compatible with the description of the boundary in Proposition 18.23
- (3).
- (3) The restriction of ev_0^∂ to the zero set of this family of multisections is a submersion.
- (4) It is compatible with forgetful map of the boundary marked points.
- (5) It is invariant under the permutation of the interior marked points.
- (6) It is invariant under the cyclic permutation of the boundary marked points.

The proof is the same as that of Lemma 18.14 and is omitted. We now define

$$q_{\ell,k;\beta}^{F,S \geq 0} : E_\ell(\Omega(M)[2]) \otimes \Omega(M)[1] \otimes B_k(\Omega(L)[1]) \rightarrow \Omega(L)[1] \quad (18.30)$$

by sending $(g_1, \dots, g_\ell; h; h_1, \dots, h_k)$ to

$$ev_{0!}^\partial (ev_1^* g_1 \wedge \dots \wedge ev_\ell^* g_\ell \wedge ev_{-\infty}^* h \wedge ev_1^{\partial*} h_1 \wedge \dots \wedge ev_k^{\partial*} h_k).$$

Here

$$\begin{aligned} (ev, ev^\partial, ev_{-\infty}) &= (ev_1, \dots, ev_\ell; ev^\partial; ev_0^\partial, \dots, ev_k^\partial) \\ &: \mathcal{M}_{k+1;\ell}(\text{para}; F, J; *, L; \beta) \rightarrow M^{\ell+1} \times L^{k+1} \end{aligned} \quad (18.31)$$

is the natural evaluation map and $ev_{0!}^\partial$ is the integration along the fibers of ev_0^∂ via the correspondence given by this moduli space. We define

$$q_{\ell,k;\beta}^{F,S=0} : E_\ell(\Omega(M)[2]) \otimes \Omega(M)[1] \otimes B_k(\Omega(L)[1]) \rightarrow \Omega(L)[1]$$

by using $\mathcal{M}_{k+1;\ell}(F_0, J; *, L; \beta)$ in (18.28) in the same way.

Definition 18.25. Define a map

$$\mathfrak{J}_{(F_0,J)}^{\mathbf{b},\mathbf{b}} : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow CF_{\text{dR}}(L; \Lambda^\downarrow)$$

by

$$\mathfrak{J}_{(F_0,J)}^{\mathbf{b},\mathbf{b}}(h) = \sum_{\beta} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} q^{-\beta \cap \omega} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{\ell!} q_{\ell,k;\beta}^{F,S=0} (\mathbf{b}_+^{\otimes \ell}, h, b_+^{\otimes k}). \quad (18.32)$$

We also define

$$\mathfrak{H}_{(F,J)}^{\mathbf{b},\mathbf{b}} : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow CF_{\text{dR}}(L; \Lambda^\downarrow)$$

by

$$\mathfrak{H}_{(F,J)}^{\mathbf{b},\mathbf{b}}(h) = \sum_{\beta} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} q^{-\beta \cap \omega} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{\ell!} q_{\ell,k;\beta}^{F,S \geq 0} (\mathbf{b}_+^{\otimes \ell}, h, b_+^{\otimes k}). \quad (18.33)$$

Lemma 18.26. *We have*

$$\delta^{\mathbf{b}} \circ \mathfrak{H}_{(F,J)}^{\mathbf{b},\mathbf{b}} \pm \mathfrak{H}_{(F,J)}^{\mathbf{b},\mathbf{b}} \circ d = \mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}} \circ \mathcal{P}_{(H_\chi,J)}^{\mathbf{b}} \circ \flat - \mathfrak{J}_{(F_0,J)}^{\mathbf{b},\mathbf{b}}.$$

Here $\flat : \Omega^*(M) \hat{\otimes} \Lambda^\downarrow \cong \Omega_{\dim M-*}(M) \hat{\otimes} \Lambda^\downarrow$ as in (3.19).

Proof. This follows from Lemma 18.23 after considering the correspondence by using the moduli space in Lemma 18.23 (1). Indeed, the first term of left hand side corresponds to (18.27). The first and second terms of the right hand side correspond to (18.26) and (18.28), respectively. \square

We next construct a chain homotopy between $\mathfrak{J}_{(F_0,J)}^{\mathbf{b},\mathbf{b}}$ and $i_{\text{qm},\mathbf{b}}$. Let $\sigma \in [0, 1]$. We replace (18.24) by

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - \sigma X_{F_0}(u) \right) = 0 \quad (18.34)$$

in Definition 18.22 to define $\mathring{\mathcal{M}}_{k+1;\ell}(\sigma, F_0, J; *, L; \beta)$. We put

$$\mathring{\mathcal{M}}_{k+1;\ell}([0, 1], F_0, J; *, L; \beta) = \bigcup_{\sigma \in [0, 1]} \{\sigma\} \times \mathring{\mathcal{M}}_{k+1;\ell}(\sigma, F_0, J; *, L; \beta).$$

We can prove a lemma similar to Lemmas 18.23, 18.24 using the compactification $\mathring{\mathcal{M}}_{k+1;\ell}([0, 1], F_0, J; *, L; \beta)$ in place of $\mathcal{M}_{k+1;\ell}(\text{para}, F, J; *, L; \beta)$ in (18.31) and (18.33), and define

$$\overline{\mathfrak{H}}_{(F_0,J)}^{\mathbf{b},\mathbf{b}} : \Omega(M) \hat{\otimes} \Lambda^\downarrow \rightarrow CF_{\text{dR}}(L; \Lambda^\downarrow).$$

Then in a similar way we can show

$$\delta^{\mathbf{b}} \circ \overline{\mathfrak{H}}_{(F_0,J)}^{\mathbf{b},\mathbf{b}} \pm \overline{\mathfrak{H}}_{(F_0,J)}^{\mathbf{b},\mathbf{b}} \circ d = \mathfrak{J}_{(F_0,J)}^{\mathbf{b},\mathbf{b}} - i_{\text{qm},\mathbf{b}}.$$

Combining this with Lemma 18.4, we finish the proof of Proposition 18.21. \square

18.5. Heavyness of L . We are now ready to complete the proofs of Proposition 18.9 and Theorem 18.8.

Proof of Proposition 18.9. For $\epsilon > 0$ we take $x \in F^\lambda CF(M, H; \Lambda^\downarrow) \cong F^{-\lambda} CF(M, H; \Lambda)$ such that $[x] = \mathcal{P}_{(H_\chi,J),*}^{\mathbf{b}}(a^\flat)$ and $\lambda \leq \rho^{\mathbf{b}}(H; a) + \epsilon$. By Proposition 18.19 we have

$$\mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}}(x) \in F^{\lambda+R+\epsilon} CF(L; \Lambda^\downarrow). \quad (18.35)$$

On the other hand, Proposition 18.21 shows that

$$[\mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b}}(x)] = \mathfrak{J}_{(F,J)}^{\mathbf{b},\mathbf{b},*} \circ \mathcal{P}_{(H_\chi,J),*}^{\mathbf{b}}(a^\flat) = i_{\text{qm},\mathbf{b}}^*(a).$$

Therefore

$$\lambda + R + \epsilon \geq \rho_L^{\mathbf{b}}(a)$$

by definition. It implies

$$\rho^{\mathbf{b}}(H; a) + 2\epsilon \geq \rho_L^{\mathbf{b}}(a) - R.$$

The proof of Proposition 18.9 is complete. \square

Proof of Theorem 18.8. We first prove (1). By Remark 18.6 it suffices to prove μ_e^b -heaviness. Let $H : M \times S^1 \rightarrow \mathbb{R}$ be a normalized periodic Hamiltonian. We put

$$H_{(n)}(t, x) = nH(nt - [nt], x), \quad (18.36)$$

where $[c]$ is the largest integer such that $c \geq [c]$. It is easy to see that $\tilde{\psi}_{H_{(n)}} = (\tilde{\psi}_H)^n$. We apply Proposition 18.9 to $H_{(n)}$ and obtain

$$\rho^b((\tilde{\psi}_H)^n; e) \geq n \inf\{-H(t, x) \mid t \in S^1, x \in L\} + \rho_L^b(e).$$

Therefore by definition we have

$$\mu_e^b(\tilde{\psi}_H) \geq \text{vol}_\omega(M) \inf\{-H(t, x) \mid t \in S^1, x \in L\}.$$

Thus Theorem 18.8 (1) is proved.

We turn to the proof of (2). Again it suffices to prove μ_e^b -superheaviness.

We use our assumption to apply Lemma 16.5 and obtain

$$\rho^b((\tilde{\psi}_H)^n; e) \leq 3\mathfrak{v}_q(e) - \rho^b((\tilde{\psi}_H)^{-n}; e). \quad (18.37)$$

We put $\tilde{H}(t, x) = -H(1 - t, x)$ and then obtain $\tilde{H}_{(n)}$ as in (18.36). We then apply Proposition 18.9 to $\tilde{H}_{(n)}$ and obtain

$$\rho^b((\tilde{\psi}_H)^{-n}; e) \geq -n \sup\{-H(t, x) \mid t \in S^1, x \in L\} + \rho_L^b(e). \quad (18.38)$$

By (18.37) and (18.38) we have

$$\rho^b((\tilde{\psi}_H)^n; e) \leq n \sup\{-H(t, x) \mid t \in S^1, x \in L\} + 3\mathfrak{v}_q(e) - \rho_L^b(e).$$

Therefore

$$\mu_e^b(\tilde{\psi}_H) \leq \text{vol}_\omega(M) \sup\{-H(t, x) \mid t \in S^1, x \in L\}.$$

as required. The proof of Theorem 18.8 is now complete. \square

19. LINEAR INDEPENDENCE OF QUASIMORPHISMS.

In this section we prove Corollary 1.10. We use the same notations as those in this corollary. Let $U_i \subset M$, $i = -N, \dots, N$ be open sets such that $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$ and $L_i \subset U_i$ for $i = 1, \dots, N$. For $i = -N, \dots, N$, let ρ_i be nonnegative smooth functions on M such that $\text{supp } \rho_i \subset U_i$ ($i = -N, \dots, N$), $\rho_i \equiv 1$ on L_i ($i = 1, \dots, N$), and $\int_M \rho_i \omega^n = c$ where $c > 0$ is independent of i ($i = 1, \dots, N$). We then put

$$H_i = \text{vol}_\omega(M)^{-1} (\rho_i - \rho_{-i})$$

and regard them as time independent normalized Hamiltonian functions. We put $\tilde{\psi}_i = \tilde{\psi}_{H_i}$. Since the support of H_i is disjoint from that of H_j it follows that $\tilde{\psi}_i$ for $i \neq j$ commutes with $\tilde{\psi}_j$. Namely they generate a subgroup isomorphic to \mathbb{Z}^N .

For $(k_1, \dots, k_N) \in \mathbb{Z}^N$ we consider

$$\tilde{\phi} = \prod_{i=1}^N \tilde{\psi}_i^{k_i}.$$

Note that $\tilde{\phi} = \tilde{\psi}_H$ where $H = \sum_{i=1}^N k_i H_i$. Since L_i is $\mu_{e_i}^{b_i}$ -superheavy and $\mu_{e_i}^{b_i}$ -heavy, we have

$$\text{vol}_\omega(M) \inf\{-H(x) \mid x \in L_i\} \leq \mu_{e_i}^{b_i}(\tilde{\phi}) \leq \text{vol}_\omega(M) \sup\{-H(x) \mid x \in L_i\}.$$

Therefore $\mu_{e_i}^{b_i}(\tilde{\phi}) = -k_i$. The proof of Corollary 1.10 is complete. \square

Part 5. Applications

In this chapter, we provide applications of the results obtained in the previous chapters. Especially combining them with the calculations we carried out in a series of papers [FOOO2, FOOO3, FOOO6] in the case of toric manifolds, we prove Corollary 1.4, and Theorem 1.11 for the case of k (≥ 2) points blow up of $\mathbb{C}P^2$. The latter example has been studied in [FOOO3]. We also examine a continuum of Lagrangian tori in $S^2 \times S^2$ discovered by the present authors in [FOOO5] and prove Theorem 1.11.

20. LAGRANGIAN FLOER THEORY OF TORIC FIBERS: REVIEW

20.1. Toric manifolds: review. In this subsection we review a very small portion of the theory of toric variety. See for example [Ful] for a detailed account of toric varieties.

Let (M, ω, J) be a Kähler manifold, where J is its complex structure and ω its Kähler form. Let n be the complex dimension of M . We assume n dimensional real torus $T^n = (S^1)^n$ acts effectively on M such that J and ω are preserved by the action. We call such (M, ω, J) a *Kähler toric manifold* if the T^n action has a moment map in the sense we describe below. Hereafter we simply say (M, ω, J) (or M) is a toric manifold.

Let (M, ω, J) be as above. We say a map $\pi = (\pi_1, \dots, \pi_n) : M \rightarrow \mathbb{R}^n$ is a *moment map* if the following holds. We consider the i -th factor S_i^1 of T^n . (Here $i = 1, \dots, n$.) Then $\pi_i : M \rightarrow \mathbb{R}$ is the moment map of the action of S_i^1 . In other words, we have the following identity of π_i

$$2\pi d\pi_i = \omega(\cdot, \tilde{\mathbf{t}}_i) \quad (20.1)$$

where $\tilde{\mathbf{t}}_i$ is the Killing vector field associated to the action of the circle S_i^1 on X .

Remark 20.1. We put 2π in Formula (20.1) in order to eliminate this factor from (20.5). See Remark 20.3.

Let $\mathbf{u} \in \text{Int}P$. Then the inverse image $\pi^{-1}(\mathbf{u})$ is a Lagrangian submanifold which is an orbit of the T^n action. We put

$$L(\mathbf{u}) = \pi^{-1}(\mathbf{u}). \quad (20.2)$$

This is a Lagrangian torus.

It is well-known that $P = \pi(M)$ is a convex polytope. We can find a finitely many affine functions $\ell_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ($j = 1, \dots, m$) such that

$$P = \{\mathbf{u} \in \mathbb{R}^n \mid \ell_j(\mathbf{u}) \geq 0, \quad \forall j = 1, \dots, m\}. \quad (20.3)$$

We put $\partial_j P = \{\mathbf{u} \in P \mid \ell_j(\mathbf{u}) = 0\}$ and $D_j = \pi^{-1}(\partial_j P)$. ($\dim_{\mathbb{R}} \partial_j P = n - 1$.) $D_1 \cup \dots \cup D_m$ is called the *toric divisor*.

Moreover we may choose ℓ_j so that the following holds.

Condition 20.2. (1) We put

$$d\ell_j = \vec{v}_j = (v_{j,1}, \dots, v_{j,n}) \in \mathbb{R}^n.$$

Then $v_{j,i} \in \mathbb{Z}$.

(2) Let p be a vertex of P . Then the number of faces $\partial_j P$ which contain p is n . Let $\partial_{j_1} P, \dots, \partial_{j_n} P$ be those faces. Then $\vec{v}_{j_1}, \dots, \vec{v}_{j_n}$ (which is contained in \mathbb{Z}^n by item (1)) is a basis of \mathbb{Z}^n .

The affine functions ℓ_j have the following geometric interpretation. Let $\mathbf{u} \in \text{Int}P$. There exists m elements $\beta_j \in H_2(M, L(\mathbf{u}); \mathbb{Z})$ such that

$$\beta_j \cap D_{j'} = \begin{cases} 1 & j = j', \\ 0 & j \neq j'. \end{cases} \quad (20.4)$$

The existence of such ℓ_j and the property above is proved in [Gu] Theorem 4.5. Then the following area formula

$$\int_{\beta_j} \omega = \ell_j(\mathbf{u}) \quad (20.5)$$

is proved in [CO] Theorem 8.1. (See [FOOO2] Section 2 also.)

Remark 20.3. Note in [CO] Theorem 8.1, [FOOO2] Section 2 there is a factor 2π in the right hand side of (20.5). We eliminate it by slightly changing the notation of moment map (See Remark 20.1.) Note in [FOOO2] the constibution of the pseudo-holomorphic disc of homology class β in \mathfrak{m}_k has weight $T^{\beta \cap \omega / 2\pi}$. In this paper and in [FOOO1] the weight is $T^{\beta \cap \omega}$.

20.2. Review of Floer cohomology of toric fiber. Let $\widehat{\mathcal{H}}^{2k}$ be the \mathbb{C} vector space whose basis is a complex codimension k submanifold of M which arises as a transversal intersection of k irreducible components D_{j_1}, \dots, D_{j_k} of the toric divisor. For $k = 0$ we let $\widehat{\mathcal{H}}^0 = \mathbb{C}$ and its basis is regarded as a codimension 0 submanifold M itself. For $k \neq 0$ the inclusion map induces an isomorphism

$$\widehat{\mathcal{H}}^{2k} \cong H_{2n-2k}(M \setminus L(\mathbf{u}); \mathbb{C}) \cong H^{2k}(M, L(\mathbf{u}); \mathbb{C}). \quad (20.6)$$

There exists a short exact sequence

$$0 \rightarrow H_{2n-2k}(M; \mathbb{Z}) \rightarrow H_{2n-2k}(M, L(\mathbf{u}); \mathbb{Z}) \rightarrow H_{2n-2k-1}(L(\mathbf{u}); \mathbb{Z}) \rightarrow 0. \quad (20.7)$$

Note that $L(\mathbf{u})$ is a torus and so $H(L(\mathbf{u}); \mathbb{Z})$ is a free abelian group. We fix a splitting of (20.7) and identify

$$H_2(M, L(\mathbf{u}); \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \oplus H_1(L(\mathbf{u}); \mathbb{Z}). \quad (20.8)$$

For $k \neq 0$, we also fix a \mathbb{C} linear subspace $\mathcal{H}^{2k} \subset \widehat{\mathcal{H}}^{2k}$ such that the homomorphism induced by the inclusion $H_{2n-2k}(M \setminus L(\mathbf{u}); \mathbb{Z}) \rightarrow H_{2n-2k}(M; \mathbb{Z})$ restricts to an isomorphism from $\widehat{\mathcal{H}}^{2k}$ to $H_{2n-2k}(M; \mathbb{Z})$. For $k = 0$ we have an isomorphism

$$\mathcal{H}^0 \cong \mathbb{C} \cong H^0(M; \mathbb{C}),$$

whose basis is the canonical unit of $H^0(M; \mathbb{C})$. We note that the odd degree cohomology of toric manifolds are all trivial.

We put $\mathcal{H} = \bigoplus_{k=0}^n \mathcal{H}^{2k}$. We take its basis $\{PD([D_a]) \mid a = 0, \dots, B\}$ so that $D_0 = [M]$ (whose Poincaré dual is the unit), each of D_1, \dots, D_{B_2} is an irreducible component of the toric divisor ($B_2 = \text{rank } H_2(M; \mathbb{Q})$) and D_{B_2+1}, \dots, D_B are transversal intersection of irreducible components of the toric divisors. ($B+1 = \text{rank } H(M; \mathbb{Q})$.) We put $\mathbf{e}_a^M = PD([D_a])$.

We put $\underline{B} = \{1, \dots, B\}$ and denote the set of all maps $\mathbf{p} : \{1, \dots, \ell\} \rightarrow \underline{B}$ by $\text{Map}(\ell, \underline{B})$. We write $|\mathbf{p}| = \ell$ if $\mathbf{p} \in \text{Map}(\ell, \underline{B})$.

For $k, \ell \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(M, L(\mathbf{u}); \mathbb{Z})$ we define a fiber product

$$\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}) = \mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta)_{(ev_1, \dots, ev_\ell)} \times_{M^\ell} \prod_{i=1}^{\ell} D_{\mathbf{p}(i)}, \quad (20.9)$$

where $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta)$ is a moduli space defined in Definition 17.2 and Proposition 17.3.

Let \mathfrak{S}_ℓ be the symmetric group of order $\ell!$. It acts on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta)$ as the permutation of the interior marked points. We define $\sigma \cdot \mathbf{p} = \mathbf{p} \circ \sigma^{-1}$. They induce a map $\sigma_* : \mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p}) \rightarrow \mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \sigma \cdot \mathbf{p})$.

Since $L(\mathbf{u})$ is a T^n orbit, $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p})$ has a T^n action induced by one on M . To describe the boundary of $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p})$ we need a notation. We will define a map

$$\text{Split} : \text{Shuff}(\ell) \times \text{Map}(\ell, \underline{B}) \longrightarrow \bigcup_{\ell_1 + \ell_2 = \ell} \text{Map}(\ell_1, \underline{B}) \times \text{Map}(\ell_2, \underline{B}), \quad (20.10)$$

as follows: Let $\mathbf{p} \in \text{Map}(\ell, \underline{B})$ and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$. We put $\ell_j = \#(\mathbb{L}_j)$ and let $\mathbf{i}_j : \{1, \dots, \ell_j\} \cong \mathbb{L}_j$ be the order preserving bijection. We consider the map $\mathbf{p}_j : \{1, \dots, \ell_j\} \rightarrow \underline{B}$ defined by $\mathbf{p}_j(i) = \mathbf{p}(\mathbf{i}_j(i))$, and set

$$\text{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p}) := (\mathbf{p}_1, \mathbf{p}_2).$$

Lemma 20.4. (1) $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p})$ has a Kuranishi structure with corners.
 (2) The Kuranishi structure is invariant under the T^n action.
 (3) Its boundary is described by the union of fiber products:

$$\mathcal{M}_{k_1+1;\#\mathbb{L}_1}(L(\mathbf{u});\beta_1; \mathbf{p}_1)_{\text{ev}_0^\partial} \times_{\text{ev}_i^\partial} \mathcal{M}_{k_2+1;\#\mathbb{L}_2}(L(\mathbf{u});\beta_2; \mathbf{p}_2) \quad (20.11)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, k_1, k_2 with $k_1 + k_2 = k$ and $\beta_1, \beta_2 \in H_2(M, L(\mathbf{u}); \mathbb{Z})$ with $\beta = \beta_1 + \beta_2$. We put $\text{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p}) = (\mathbf{p}_1, \mathbf{p}_2)$.

(4) The dimension is

$$\dim \mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p}) = n + \mu_{L(\mathbf{u})}(\beta) + k - 2 + 2\ell - \sum_{i=1}^{\ell} 2 \deg D_{\mathbf{p}(i)}. \quad (20.12)$$

- (5) The evaluation maps ev_i^∂ at the boundary marked points of $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta)$ define a strongly continuous smooth map on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p})$, which we denote by ev_i^∂ also. It is compatible with (3).
 (6) We can define an orientation of the Kuranishi structure so that it is compatible with (3).
 (7) ev_0^∂ is weakly submersive.
 (8) The Kuranishi structure is compatible with the action of the symmetry group \mathfrak{S}_ℓ .
 (9) The Kuranishi structure is compatible with the forgetful map of the i -th boundary marked point for $i = 1, \dots, k$. (We do not require the compatibility with the forgetful map of the 0-th marked point.)

Lemma 20.4 is proved in [FOOO3] Section 6.

Lemma 20.5. There exists a system of multisections on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u});\beta; \mathbf{p})$ with the following properties:

- (1) They are transversal to 0.
 (2) They are invariant under the T^n action.
 (3) They are compatible with the description of the boundary in Lemma 20.4 (3).
 (4) The restriction of ev_0^∂ to the zero set of this multisection is a submersion.
 (5) They are invariant under the action of \mathfrak{S}_ℓ .

- (6) *The multisection is compatible with the forgetful map of the i -th boundary marked point for $i = 1, \dots, k$.*

This is also proved in [FOOO3] Section 6. We note that (4) is a consequence of (2).

Let $h_1, \dots, h_k \in \Omega(L(\mathbf{u}))$. We then define a differential form on $L(\mathbf{u})$ by

$$\mathfrak{q}_{\ell,k;\beta}^T(\mathbf{p}; h_1, \dots, h_k) = (\mathrm{ev}_0^\partial)_! (\mathrm{ev}_1^\partial, \dots, \mathrm{ev}_k^\partial)^* (h_1 \wedge \dots \wedge h_k), \quad (20.13)$$

where we use the evaluation map

$$(\mathrm{ev}_0^\partial, \dots, \mathrm{ev}_k^\partial) : \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}) \rightarrow L(\mathbf{u})^{k+1}$$

and $(\mathrm{ev}_0)_!$ is the integration along the fiber. Here the superscript T stands for T^n equivariance. By Lemma 20.5 (4) integration along fiber is well-defined. By Lemma 20.5 (5), the operation (20.13) is invariant under permutation of the factors of \mathbf{p} . Therefore by the \mathbb{C} linearity, we define

$$\mathfrak{q}_{\ell,k;\beta}^T : E_\ell(\mathcal{H}[2]) \otimes B_k(\Omega(L(\mathbf{u}))[1]) \rightarrow \Omega(L(\mathbf{u}))[1]. \quad (20.14)$$

We identify the de Rham cohomology group $H(L(\mathbf{u}); \mathbb{C})$ of $L(\mathbf{u})$ with the set of T^n invariant differential forms on $L(\mathbf{u})$. By Lemma 20.5 (2), the operations $\mathfrak{q}_{\ell,k;\beta}^T$ induce

$$\mathfrak{q}_{\ell,k;\beta}^T : E_\ell(\mathcal{H}[2]) \otimes B_k(H(L(\mathbf{u}); \mathbb{C}))[1] \rightarrow H(L(\mathbf{u}); \mathbb{C})[1]. \quad (20.15)$$

In the case $\beta = \beta_0 = 0$, we define $\mathfrak{q}_{0,k;\beta_0}^T$ by (18.18). The operators $\mathfrak{q}_{\ell,k;\beta}^T$ satisfy the conclusion of Theorem 17.1. We use it in the same way as in Definition 17.7 to define $\mathfrak{m}_k^{T,\mathbf{b}}$ for $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+)$. We have thus obtained a filtered A_∞ algebra $(CF_{\mathrm{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{T,\mathbf{b}}\}_{k=0}^\infty)$ with

$$CF_{\mathrm{dR}}(L(\mathbf{u}); \Lambda_0) = \Omega(L(\mathbf{u})) \hat{\otimes} \Lambda_0.$$

This is the filtered A_∞ algebra we use in [FOOO3]. (In [FOOO3] \mathfrak{q}^T is denoted by \mathfrak{q}^{dR} .) In particular, if $\mathfrak{m}_0^{T,\mathbf{b}}(1) \equiv 0 \pmod{\mathbf{e}_L \Lambda_+}$, we have $\mathfrak{m}_1^{T,\mathbf{b}} \circ \mathfrak{m}_1^{T,\mathbf{b}} = 0$. We put $\delta^{T,\mathbf{b}} = \mathfrak{m}_1^{T,\mathbf{b}}$ and define

$$HF_T((L, \mathbf{b}); \Lambda_0) = \frac{\mathrm{Ker} \delta^{T,\mathbf{b}}}{\mathrm{Im} \delta^{T,\mathbf{b}}}. \quad (20.16)$$

We put subscript T in the notation to indicate that we are using a T^n -equivariant perturbation. In a series of papers [FOOO2, FOOO3, FOOO6] we studied Floer cohomology (20.16) and described its nonvanishing property in terms of the critical point theory of certain non-Archimedean analytic function, called the *potential function*. Explanation of this potential function is in Subsection 20.4.

20.3. Relationship with the Floer cohomology in Section 17. To apply the Floer cohomology (20.16) for the purpose of studying spectral invariants, we need to show that (20.16) is isomorphic to the Floer cohomology we used in Chapter 3. We use the next proposition for this purpose. We denote $\mathbf{b}^{(0)} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, 0)$ as before.

Proposition 20.6. *The filtered A_∞ algebra $(CF_{\mathrm{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{\mathbf{b}^{(0)}}\}_{k=0}^\infty)$ is homotopy equivalent to $(CF_{\mathrm{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{T,\mathbf{b}^{(0)}}\}_{k=0}^\infty)$ as a unital filtered A_∞ algebra.*

Here the first filtered A_∞ algebra is defined in Definition 17.7 and the second one is defined at the end of Subsection 20.2. Proposition 20.6 is the de Rham version of the filtered A_∞ algebra associated to a Lagrangian submanifold, which was established in [FOOO1] Theorem A. Since the details of the construction we are using here is slightly different from those in [FOOO1], we give a proof of Proposition 20.6 in Section 28 for completeness' sake.

Let $\mathbf{e}_L = 1$ be the differential 0-form on L which is the unit of our filtered A_∞ algebra. We put:

$$\begin{aligned} & \widehat{\mathcal{M}}_{\text{weak,def}}(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)}) \\ &= \left\{ b_+ \in H^{\text{odd}}(L(\mathbf{u}); \Lambda_+) \mid \sum_{k=0}^{\infty} \mathbf{m}_k^{\mathbf{b}^{(0)}}(b_+^k) \equiv 0 \pmod{\mathbf{e}_L \Lambda_+} \right\} \end{aligned} \quad (20.17)$$

and

$$\begin{aligned} & \widehat{\mathcal{M}}_{\text{weak,def}}^T(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)}) \\ &= \left\{ b_+ \in H^{\text{odd}}(L(\mathbf{u}); \Lambda_+) \mid \sum_{k=0}^{\infty} \mathbf{m}_k^{T, \mathbf{b}^{(0)}}(b_+^k) \equiv 0 \pmod{\mathbf{e}_L \Lambda_+} \right\}. \end{aligned} \quad (20.18)$$

We write $\mathbf{b}(b_+) = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+)$. Then

$$\mathbf{b}(b_+) \in \widehat{\mathcal{M}}_{\text{weak,def}}(L(\mathbf{u}); \Lambda_0)$$

for $b_+ \in \widehat{\mathcal{M}}_{\text{weak,def}}(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)})$.

Similar fact holds for $b_+ \in \widehat{\mathcal{M}}_{\text{weak,def}}^T(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)})$.

Proposition 20.6 and the homotopy theory of filtered A_∞ algebras as given in [FOOO1] Chapter 4 immediately imply the following:

Corollary 20.7. *There exists a map*

$$\mathfrak{J}_* : \widehat{\mathcal{M}}_{\text{weak,def}}^T(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)}) \rightarrow \widehat{\mathcal{M}}_{\text{weak,def}}(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)})$$

such that for each $b_+ \in \widehat{\mathcal{M}}_{\text{weak,def}}^T(L(\mathbf{u}); \Lambda_+; \mathbf{b}^{(0)})$ there exists a chain homotopy equivalence

$$\mathfrak{J}_*^{b_+} : (CF(L(\mathbf{u}); \Lambda), \delta^{T, \mathbf{b}(b_+)}) \rightarrow (CF(L(\mathbf{u}); \Lambda), \delta^{\mathbf{b}(\mathfrak{J}_*(b_+))})$$

that preserves the filtration.

We note that \mathfrak{J}_* induces an isomorphism after taking gauge equivalence. We do not use this fact in this paper.

We next use $\mathbf{q}_{\ell, k; \beta}^T$ in place of $\mathbf{q}_{\ell, k; \beta}$ in (17.17) to define a chain map

$$i_{\text{qm}, \mathbf{b}}^T : (\Omega(M) \widehat{\otimes} \Lambda, d) \rightarrow (CF(L(\mathbf{u}); \Lambda), \delta^{T, \mathbf{b}}).$$

Lemma 20.8. *$\mathfrak{J}_*^{b_+} \circ i_{\text{qm}, \mathbf{b}}^T$ is chain homotopic to $i_{\text{qm}, \mathbf{b}}^T$.*

The proof is parallel to Proposition 20.6 and is given in Section 28.

Now the following is an immediate consequence.

Corollary 20.9. *When we replace $HF((L, \mathbf{b}); \Lambda)$ by $HF_T((L, \mathbf{b}); \Lambda)$ and i_{qm}^* by $i_{\text{qm}}^{T, *}$ respectively, Theorem 18.8 holds.*

20.4. Properties of Floer cohomology $HF_T((L, \mathbf{b}); \Lambda)$: review. We now go back to the study of Floer cohomology $HF_T((L, \mathbf{b}); \Lambda)$ which was established in [FOOO2, FOOO3] and which we also reviewed in Subsection 20.2.

Proposition 20.10. *If $b_+ \in H^1(L(\mathbf{u}); \Lambda_+)$, then*

$$\mathfrak{m}_0^{T, \mathbf{b}(b_+)}(1) \equiv 0 \pmod{\mathbf{e}_L \Lambda_+}.$$

This is nothing but [FOOO2] Proposition 4.3 and [FOOO3] Proposition 3.1. We omit its proof and refer readers the above references for the details. The proof is based on a dimension counting argument. We remark that the proof of Proposition 20.10 does not work if we replace $\mathfrak{m}_0^{T, \mathbf{b}(b_+)}$ by $\mathfrak{m}_0^{\mathbf{b}(b_+)}$. This is because we used a *continuous family* of multisections for the definition of $\mathfrak{m}_k^{\mathbf{b}(b_+)}$ in Section 17. So the above mentioned dimension counting argument cannot apply. (See [FOOO6] Remark 18.2.) Actually this is *the* reason why we use cycles D_a instead of differential forms to represent cohomology classes of M in [FOOO2, FOOO3] and in this section.

Now we consider $\mathbf{b}^{(0)} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, 0)$ and $b_+ \in H^1(L(\mathbf{u}); \Lambda_+)$. By Proposition 20.10 we have

$$\mathbf{b}(b_+) \in \widehat{\mathcal{M}}_{\text{weak, def}}^T(L(\mathbf{u}); \Lambda_0). \quad (20.19)$$

We use the identification (20.8) to regard $\mathbf{b}(b_+) = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+)$ as an element of $H^{\text{even}}(M; \Lambda_0) \oplus H^1(L(\mathbf{u}); \Lambda_0)$. So hereafter we define $HF_T((L(\mathbf{u}), (\mathbf{b}, b)); \Lambda_0)$ for $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0)$ and $b \in H^1(L(\mathbf{u}); \Lambda_0)$. This is the Floer cohomology we studied in [FOOO3].

We define

$$\mathfrak{P}\mathfrak{D}^{\mathbf{u}} : H^{\text{even}}(M; \Lambda_0) \times H^1(L(\mathbf{u}); \Lambda_0) \rightarrow \Lambda_+$$

by

$$\mathfrak{m}_0^{T, (\mathbf{b}, b)}(1) = \mathfrak{P}\mathfrak{D}^{\mathbf{u}}(\mathbf{b}, b) \mathbf{e}_L. \quad (20.20)$$

We now review the results of [FOOO2, FOOO3] on the potential function $\mathfrak{P}\mathfrak{D}^{\mathbf{u}}$ and how the Floer cohomology is related to it.

We fix a basis $\{\mathbf{e}_i\}_{i=1}^n$ of $H^1(L(\mathbf{u}); \mathbb{Z})$. Identifying $L(\mathbf{u})$ with T^n by the action, we can find a basis $\{\mathbf{e}_i\}_{i=1}^n$ for all $\mathbf{u} \in P$ in a canonical way. Let $b \in H^1(L(\mathbf{u}); \Lambda_0)$ we write it as

$$b = \sum x_i^{\mathbf{u}} \mathbf{e}_i \quad (20.21)$$

where $x_i^{\mathbf{u}} \in \Lambda_0$. Thus $(x_1^{\mathbf{u}}, \dots, x_n^{\mathbf{u}})$ is a coordinate of $H^1(L(\mathbf{u}); \Lambda_0)$. (To specify that it is associated with $L(\mathbf{u})$ we put \mathbf{u} in the expression $x_i^{\mathbf{u}}$ above.) Let $x_i^{\mathbf{u}} = x_{i,0}^{\mathbf{u}} + x_{i,+}^{\mathbf{u}}$ where $x_{i,0}^{\mathbf{u}} \in \mathbb{C}$ and $x_{i,+}^{\mathbf{u}} \in \Lambda_+$. We put

$$y_i^{\mathbf{u}} = \exp(x_{i,0}^{\mathbf{u}}) \exp(x_{i,+}^{\mathbf{u}}) \in \Lambda_0 \setminus \Lambda_+. \quad (20.22)$$

We note that $\exp(x_{i,0}^{\mathbf{u}}) \in \mathbb{C} \setminus \{0\}$ makes sense in the usual Archimedean sense, and

$$\exp(x_{i,+}^{\mathbf{u}}) = \sum_{k=0}^{\infty} (x_{i,+}^{\mathbf{u}})^k / k!$$

converges in T -adic topology.

Let S_i^1 be the i -th factor of T^n which corresponds to the basis element \mathbf{e}_i . We choose our moment map $\pi : M \rightarrow \mathbb{R}^n$ so that its i -th component is the moment map of the S_i^1 action. In this way we fix the coordinate of the affine space \mathbb{R}^n which contains P . Note that there is still a freedom to choose the origin $\mathbf{0} \in \mathbb{R}^n$. We do not specify this choice since it does not affect the story.

Let $\mathbf{u} = (u_1, \dots, u_n) \in P$. We put

$$y_i = T^{u_i} y_i^{\mathbf{u}}. \quad (20.23)$$

We do not put \mathbf{u} in the notation y_i above. This is justified by Theorem 20.14.

Remark 20.11. For the notational convenience we assume $\mathbf{0} \in P$. Then we will have $y_i = y_i^{\mathbf{0}}$.

With respect to the above coordinates, we may regard $\mathfrak{P}\mathfrak{O}^{\mathbf{u}}$ as a function on

$$\mathfrak{P}\mathfrak{O}^{\mathbf{u}}(\mathfrak{b}; b) = \mathfrak{P}\mathfrak{O}_{\mathfrak{b}}^{\mathbf{u}}(x_1^{\mathbf{u}}, \dots, x_n^{\mathbf{u}})$$

where $x_k^{\mathbf{u}}$ ($k = 1, \dots, n$) are the variables defined as in (20.21).

As we will see in Theorem 20.14, $\mathfrak{P}\mathfrak{O}_{\mathfrak{b}}^{\mathbf{u}}$ becomes a function of y_1, \dots, y_n and then it will be independent of \mathbf{u} . Actually it is contained in an appropriate completion of the Laurent polynomial ring $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$. Description of this completion is in order now. By (20.23), there exists an isomorphism

$$\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}] \cong \Lambda[y_1^{\mathbf{u}}, \dots, y_n^{\mathbf{u}}, (y_1^{\mathbf{u}})^{-1}, \dots, (y_n^{\mathbf{u}})^{-1}]$$

for any $\mathbf{u} \in P$. In other words any element of $\mathfrak{P} \in \Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$ can be written as a *finite* sum

$$\mathfrak{P} = \sum a_{k_1, \dots, k_n} (y_1^{\mathbf{u}})^{k_1} \dots (y_n^{\mathbf{u}})^{k_n}. \quad (20.24)$$

Note $a_{k_1, \dots, k_n} \in \Lambda$ are zero except for a finite number of them. We define

$$\mathfrak{v}_{\mathbf{u}}(\mathfrak{P}) = \min\{\mathfrak{v}_T(a_{k_1, \dots, k_n}) \mid a_{k_1, \dots, k_n} \neq 0\}. \quad (20.25)$$

This is a non-Archimedean valuation defined on $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$. We put

$$\mathfrak{v}_P(\mathfrak{P}) = \inf\{\mathfrak{v}_{\mathbf{u}}(\mathfrak{P}) \mid \mathbf{u} \in P\}.$$

This is a norm (but not a valuation) and

$$d_P(\mathfrak{P}, \mathfrak{Q}) = \exp^{-\mathfrak{v}_P(\mathfrak{P}-\mathfrak{Q})} \quad (20.26)$$

defines a metric on $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$.

For $\epsilon > 0$, denote

$$P_{\epsilon} = \{\mathbf{u} \in P \mid \forall i \ell_i(\mathbf{u}) \geq \epsilon\}.$$

We define another metric on $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$ by

$$d_{\circ}^P(\mathfrak{P}, \mathfrak{Q}) = \sum_{n=n_0}^{\infty} 2^{-n} \exp^{-\mathfrak{v}_{P_{1/n}}(\mathfrak{P}-\mathfrak{Q})}. \quad (20.27)$$

(Here we take n_0 sufficiently large so that P_{1/n_0} is nonempty.) This series obviously converges because $\mathfrak{v}_{P_{\epsilon'}} \leq \mathfrak{v}_{P_{\epsilon}}$ if $\epsilon' < \epsilon$.

Definition 20.12. We denote the completion of $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$ with respect to the metric d_P by $\Lambda\langle\langle y, y^{-1} \rangle\rangle^P$.

We denote by $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\circ P}$ the completion of $\Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}]$ with respect to the metric d_{\circ}^P .

In other words, $\Lambda\langle\langle y, y^{-1} \rangle\rangle^P$ (resp. $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\circ P}$) is the set of all \mathfrak{P} 's such that for any $\mathbf{u} \in P$ (resp. $\mathbf{u} \in \text{Int } P$) we may write \mathfrak{P} as a *possibly infinite* sum of the form (20.24) such that $\lim_{|k_1|+\dots+|k_n| \rightarrow \infty} \mathfrak{v}_T(a_{k_1, \dots, k_n}) = +\infty$.

Remark 20.13. In [FOOO3], we used a slightly different notation $\Lambda^P \langle\langle y, y^{-1} \rangle\rangle$ instead of $\Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$.

Now we have:

Theorem 20.14. *If $\mathfrak{b} \in H^{even}(M; \Lambda_0)$, then*

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}^{\mathfrak{u}} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}. \quad (20.28)$$

If $\mathfrak{b} \in H^{even}(M; \Lambda_+)$, then

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}^{\mathfrak{u}} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^P. \quad (20.29)$$

We explain the meaning of (20.28). Let $\mathfrak{P} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$ and $\mathfrak{u} \in \text{Int } P$. As we mention above \mathfrak{P} is written as a series of the form (20.24) with

$$\lim_{|k_1| + \dots + |k_n| \rightarrow \infty} \mathfrak{v}_T(a_{k_1, \dots, k_n}) = +\infty.$$

Let $b = \sum x_i^{\mathfrak{u}} \mathbf{e}_i$. Then by putting (20.22) and plugging it in (20.24) the series converges in T -adic topology and we obtain an element of Λ . Thus we obtain a function

$$\mathfrak{P}^{\mathfrak{u}} : H^1(L(\mathfrak{u}); \Lambda) \rightarrow \Lambda.$$

The statement (20.28) means that there exists $\mathfrak{P} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$ such that the above $\mathfrak{P}^{\mathfrak{u}}$ coincides with $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}^{\mathfrak{u}}$ for *any* $\mathfrak{u} \in \text{Int } P$. (We note that we require \mathfrak{P} to be *independent* of \mathfrak{u} .) The meaning of (20.29) is similar.

Actually we can show the following:

Lemma 20.15. *Let $\mathfrak{P} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$. Then \mathfrak{P} is written as a series*

$$\mathfrak{P} = \sum a_{k_1, \dots, k_n} y_1^{k_1} \dots y_n^{k_n} \quad (20.30)$$

which converges in $d_{\mathring{P}}$ topology. For any $(\mathfrak{y}_1, \dots, \mathfrak{y}_n) \in \Lambda^n$ with

$$(\mathfrak{v}_T(\mathfrak{y}_1), \dots, \mathfrak{v}_T(\mathfrak{y}_n)) \in \text{Int } P$$

the series

$$\sum a_{k_1, \dots, k_n} \mathfrak{y}_1^{k_1} \dots \mathfrak{y}_n^{k_n} \quad (20.31)$$

converges in T -adic topology.

Let $\mathfrak{P} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^P$. Then \mathfrak{P} is written as a series (20.30) which converges in d_P topology. For any $(\mathfrak{y}_1, \dots, \mathfrak{y}_n) \in \Lambda^n$ with

$$(\mathfrak{v}_T(\mathfrak{y}_1), \dots, \mathfrak{v}_T(\mathfrak{y}_n)) \in P$$

the series (20.31) converges in T -adic topology.

The proof is elementary and is omitted.

Theorem 20.14 is [FOOO3] Theorem 3.14. We do not discuss its proof in this paper but refer to [FOOO3] for the details.

We next discuss the relationship between the potential function and the non-vanishing of Floer cohomology. We first note that we can define the logarithmic derivative

$$y_i \frac{\partial \mathfrak{P}}{\partial y_i} \quad (20.32)$$

for an element \mathfrak{P} of $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$. In fact, regarding the expression (20.30) of \mathfrak{P} as a power series, we define

$$y_i \frac{\partial \mathfrak{P}}{\partial y_i} = \sum a_{k_1, \dots, k_n} k_i y_1^{k_1} \dots y_n^{k_n}.$$

It is easy to see that this series converges with respect to $d_{\mathring{P}}$ -topology and defines an element of $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$.

Definition 20.16. Let $\mathfrak{P} \in \Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$ and $\mathfrak{y} = (\mathfrak{y}_1, \dots, \mathfrak{y}_n) \in \Lambda^n$ with

$$(\mathfrak{v}_T(\mathfrak{y}_1), \dots, \mathfrak{v}_T(\mathfrak{y}_n)) \in \text{Int } P.$$

We say that \mathfrak{y} is a *critical point* of \mathfrak{P} if it satisfies

$$y_i \frac{\partial \mathfrak{P}}{\partial y_i}(\mathfrak{y}) = 0$$

for all $i = 1, \dots, n$.

For each critical point \mathfrak{y} , we define a point $\mathbf{u}(\mathfrak{y}) \in \text{Int } P$ by

$$\mathbf{u}(\mathfrak{y}) = (\mathfrak{v}_T(\mathfrak{y}_1), \dots, \mathfrak{v}_T(\mathfrak{y}_n)), \quad (20.33)$$

and an element $b = b(\mathfrak{y}) \in H^1(L(\mathbf{u}(\mathfrak{y})), \Lambda_0)$ by

$$x(\mathfrak{y})_i = \log(T^{-\mathfrak{v}_T(\mathfrak{y}_i)} y_i), \quad b(\mathfrak{y}) = \sum x(\mathfrak{y})_i e_i. \quad (20.34)$$

Here the meaning of \log in (20.34) is as follows. Note that $\mathfrak{v}_T(T^{-\mathfrak{v}_T(\mathfrak{y}_i)} y_i) = 0$. Therefore we can write

$$T^{-\mathfrak{v}_T(\mathfrak{y}_i)} y_i = c_1(1 + c_2)$$

for some $c_1 \in \mathbb{C} \setminus \{0\}$, $c_2 \in \Lambda_+$. Then we define

$$\log(T^{-\mathfrak{v}_T(\mathfrak{y}_i)} y_i) = \log c_1 + \sum_{n=1}^{\infty} (-1)^n \frac{c_2^{n+1}}{n+1}.$$

(Here we choose a branch of $\log c_1$ so that its imaginary part lies in $[0, 2\pi)$, for example.)

Theorem 20.17. Let $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0)$. If \mathfrak{y} is a critical point of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$,

$$HF((L(\mathbf{u}(\mathfrak{y})), (\mathfrak{b}, b(\mathfrak{y})); \Lambda) \cong H(T^n; \Lambda).$$

Conversely if

$$HF((L(\mathbf{u}), (\mathfrak{b}, b)); \Lambda) \neq 0,$$

there exists a critical point \mathfrak{y} of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$ such that

$$\mathbf{u} = \mathbf{u}(\mathfrak{y}), \quad b = b(\mathfrak{y}).$$

Theorem 20.17 is [FOOO3] Theorem 5.5. We refer readers to [FOOO3] for its proof.

We next describe the relation of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$ to the quantum cohomology. Consider the closed ideal of the Frechet ring $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$ generated by $\left\{ y_i \frac{\partial \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}}{\partial y_i} \mid i = 1, \dots, n \right\}$. We denote the quotient ring by

$$\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda) = \frac{\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}}{\text{Clos}_d \left(y_i \frac{\partial \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}}{\partial y_i} : i = 1, \dots, n \right)}$$

which we call the *Jacobian ring* of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$. We define a map

$$\mathfrak{k}\mathfrak{s}_{\mathfrak{b}} : H(M; \Lambda) \rightarrow \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$$

called *Kodaira-Spencer map* as follows. Let \mathbf{e}_i^M be a basis of $H(M; \mathbb{Q})$. We write an element of $H(M; \Lambda)$ as $\sum w_i \mathbf{e}_i^M$, $w_i \in \Lambda$. We may express

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}} = \sum a_{k_1, \dots, k_n}(\mathfrak{b}) y_1^{k_1} \dots y_n^{k_n},$$

where $a_{k_1, \dots, k_n}(\mathfrak{b})$ is a function of w_i (where $\mathfrak{b} = \sum w_i \mathbf{e}_i^M$). Then $a_{k_1, \dots, k_n}(\mathfrak{b})$ is a formal power series of w_i with coefficients in Λ which converges in T -adic topology. (See (20.43) for the precise description.) Therefore we can make sense of the partial derivatives $\frac{\partial a_{k_1, \dots, k_n}}{\partial w_i}$. Then we put

$$\frac{\partial \mathfrak{P}\mathfrak{D}}{\partial w_i}(\mathfrak{b}) = \sum \frac{\partial a_{k_1, \dots, k_n}}{\partial w_i}(\mathfrak{b}) y_1^{k_1} \dots y_n^{k_n}.$$

For each $\mathfrak{b} \in H(M; \Lambda_0)$, the right hand side converges and defines an element of $\Lambda \langle\langle y, y^{-1} \rangle\rangle^{\circ P}$.

Now we define the map $\mathfrak{k}\mathfrak{s}_{\mathfrak{b}}$ by setting its value to be

$$\mathfrak{k}\mathfrak{s}_{\mathfrak{b}}(\mathbf{e}_i^M) = \left[\frac{\partial \mathfrak{P}\mathfrak{D}}{\partial w_i}(\mathfrak{b}) \right]. \quad (20.35)$$

Theorem 20.18. *The map $\mathfrak{k}\mathfrak{s}_{\mathfrak{b}}$ defines a ring isomorphism*

$$(QH(M; \Lambda), \cup^{\mathfrak{b}}) \cong \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda).$$

This is [FOOO6] Theorem 1.1 for whose proof we refer readers thereto.

Remark 20.19. [FOOO6] Theorem 1.1 is stated as a result over Λ_0 -coefficients which is stronger than Theorem 20.18. We do not use this isomorphism over Λ_0 -coefficients in the present paper.

We also need a result on the structure of the Jacobian ring $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$.

Definition 20.20. We say a critical point \mathfrak{y} of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$ is *nondegenerate* if

$$\det \left[y_i y_j \frac{\partial^2 \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}}{\partial y_i \partial y_j} \right]_{i,j=1}^{i,j=n}(\mathfrak{y}) \neq 0.$$

We say $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$ is a *Morse function* if all of its critical points are nondegenerate.

Let $\text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})$ be the set of all critical points of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$.

Definition 20.21. For $\mathfrak{y} = (\mathfrak{y}_1, \dots, \mathfrak{y}_n) \in \text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})$, we define the subset $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y}) \subset \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$ as follows: Regard $y_i \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\circ P}$ and then multiplication by y_i induces an action on $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$. We denote the corresponding endomorphism by \hat{y}_i . Then we put

$$\begin{aligned} \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y}) &= \{x \in \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda) \mid (\hat{y}_i - \mathfrak{y}_i)^N x = 0, \\ &\quad \text{for all } i \text{ and sufficiently large } N\}. \end{aligned} \quad (20.36)$$

Proposition 20.22. (1) *There is a splitting of Jacobian ring*

$$\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda) \cong \prod_{\mathfrak{y} \in \text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})} \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y})$$

as a direct product of rings.

- (2) Each of $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y})$ is a local ring.
- (3) \mathfrak{y} is nondegenerate if and only if $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y}) \cong \Lambda$.

Proposition 20.22 is [FOOO6] Proposition 2.15, to which we refer readers for its proof.

It follows from Proposition 20.22 that the set of indecomposable idempotents of $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$ one-one corresponds to $\text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})$. We denote by $1_{\mathfrak{y}} \in \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y})$ the unit of the ring $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y})$ which corresponds to an idempotent of $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$. Denote by $e_{\mathfrak{y}}$ the idempotent of $(QH(M; \Lambda), \cup^{\mathfrak{b}})$ corresponding to $1_{\mathfrak{y}}$ under the isomorphism $\mathfrak{k}\mathfrak{s}_{\mathfrak{b}}$ in Theorem 20.18.

We are finally ready to describe the map

$$i_{\text{qm}, (\mathfrak{b}, b)}^{T,*} : QH_{\mathfrak{b}}^*(M; \Lambda) \rightarrow HF^*((L(\mathbf{u}), (\mathfrak{b}, b)); \Lambda) \quad (20.37)$$

in our situation.

Theorem 20.23. *Let $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0)$, \mathfrak{y} a critical point of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}$, $a \in H(M; \Lambda)$ and $\mathfrak{P} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$ such that*

$$\mathfrak{k}\mathfrak{s}_{\mathfrak{b}}(a) = [\mathfrak{P}] \mod \text{Clos}_d \left(y_i \frac{\partial \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}}{\partial y_i} : i = 1, \dots, n \right). \quad (20.38)$$

Then we have

$$i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(a) = \mathfrak{P}(\mathfrak{y})e_{L(\mathbf{u}(\mathfrak{y}))}. \quad (20.39)$$

Proof. This is [FOOO6] Lemma 17.1. Since its proof is omitted in [FOOO6], we provide its proof here.

We note that the right hand side of (20.39) is independent of the choices of \mathfrak{P} satisfying (20.38). This is because $y_i \frac{\partial \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}}{\partial y_i}$ is zero at \mathfrak{y} .

Let $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_2 + \mathfrak{b}_+$ be as in (5.5). We put $\mathfrak{b} = \sum w_i(\mathfrak{b})\mathbf{e}_i^M$ and $b = \sum y_i(b)\mathbf{e}_i = b_0 + b_+$ where $b_0 \in H^1(L(\mathbf{u}); \mathbb{C})$ and $b_+ \in H^1(L(\mathbf{u}); \Lambda_+)$. By definition, we have

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}(b) = \mathfrak{b}_0 + \sum_{\beta, k, \ell} T^{\omega \cap \beta} \frac{\exp(\mathfrak{b}_2 \cap \beta + b_0 \cap \partial \beta)}{\ell!} \mathfrak{q}_{k, \ell, \beta}^T(\mathfrak{b}_+^{\otimes \ell}, b_+^{\otimes k}), \quad (20.40)$$

where we identify $H^0(L(\mathbf{u}); \Lambda) = \Lambda$.

We further split $\mathfrak{b} = \mathfrak{b}_0 + \widehat{\mathfrak{b}}_2 + \mathfrak{b}_{\text{high}}$ so that

$$\widehat{\mathfrak{b}}_2 \in H^2(M; \Lambda_0), \quad \mathfrak{b}_{\text{high}} \in \bigoplus_{k \geq 1} H^{2k}(M; \Lambda_0).$$

By [FOOO3] Lemmas 7.1 and 9.2, we can rewrite (20.40) to

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}(b) = \mathfrak{b}_0 + \sum_{\beta, \ell} T^{\omega \cap \beta} \frac{\exp(\widehat{\mathfrak{b}}_2 \cap \beta + b \cap \partial \beta)}{\ell!} \mathfrak{q}_{0, \ell, \beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell}, 1). \quad (20.41)$$

We use the coordinate $w' = (w_{B_1+1}, \dots, w_B)$ for $\mathfrak{b}_{\text{high}}$. Namely we put $\mathfrak{b}_{\text{high}} = \sum_{i=B_1+1}^B w_i \mathbf{e}_i^M$. Then we define

$$P_{\beta}(w') = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \mathfrak{q}_{0, \ell, \beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell}, 1). \quad (20.42)$$

Lemma 20.24.

$$P_{\beta}(w') \in \Lambda[w_{B_1+1}, \dots, w_B].$$

Proof. Since each of the component of $\mathfrak{b}_{\text{high}}$ has degree 4 or higher, we can show that $\mathfrak{q}_{0,\ell,\beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell}, 1)$ is nonzero for only a finite number of ℓ , by a dimension counting. Each of $\mathfrak{q}_{0,\ell,\beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell}, 1)$ is a polynomial of w_i . Therefore $P_\beta(w')$ is a polynomial as asserted. \square

We re-enumerate D_1, D_2, \dots so that $\{D_1, \dots, D_{B_1}\}$ becomes a \mathbb{Q} -basis of $H^2(M; \mathbb{Q})$. Then w_1, \dots, w_{B_1} are the corresponding coordinates of $H^2(M; \mathbb{Q})$. We put

$$\mathfrak{w}_i = e^{w_i} = \sum_{k=0}^{\infty} \frac{w_i^k}{k!}.$$

It follows from (20.41) and (20.42) that we can write

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}} = w_0 + \sum_{\beta} T^{\omega \cap \beta} \mathfrak{w}_1^{\beta \cap D_1} \dots \mathfrak{w}_{B_1}^{\beta \cap D_{B_1}} y_1^{\partial \beta \cap e_1} \dots y_n^{\partial \beta \cap e_n} P_\beta(w'). \quad (20.43)$$

Here we regard β as an element of $H_2(M; L(\mathbf{0}); \mathbb{Z})$ to define $\omega \cap \beta$ in (20.43).

We will compare $i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(\mathbf{e}_i^M)$ with the w_i -derivative of (20.43). By definition, we have

$$\begin{aligned} & i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(\mathbf{e}_i^M) \\ &= \sum_{\beta, k, \ell_1, \ell_2} T^{\omega \cap \beta} \frac{\exp(\mathfrak{b}_2 \cap \beta + b_0 \cap \partial \beta)}{(\ell_1 + \ell_2 + 1)!} \mathfrak{q}_{k, \ell_1 + \ell_2 + 1, \beta}^T(\mathfrak{b}_+^{\otimes \ell_1} \mathbf{e}_i^M \mathfrak{b}_+^{\otimes \ell_2}, \mathfrak{b}_+^{\otimes k}). \end{aligned} \quad (20.44)$$

We consider three cases separately:

(Case 1; $i = 0$): It is easy to see that

$$\mathfrak{q}_{k, \ell, \beta}^T(\mathfrak{b}_+^{\otimes \ell_1} \mathbf{e}_0^M \mathfrak{b}_+^{\otimes \ell_2}; \mathfrak{b}_+^{\otimes k}) = 0$$

unless $\beta = 0$ and $k = \ell = 0$. Therefore we have

$$i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(\mathbf{e}_0^M) = \mathfrak{q}_{0,0,0}^T(\mathbf{e}_0^M) = \mathbf{e}_L.$$

Since

$$\frac{\partial \mathfrak{P}\mathfrak{D}_{\mathfrak{b}(w)}}{\partial w_0} = 1$$

by (20.43), we have (20.39) for $a = \mathbf{e}_0^M$.

(Case 2; $i > B_1$): By [FOOO3] Lemmas 7.1 and 9.2 we can rewrite (20.44) to

$$\begin{aligned} & i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(\mathbf{e}_i^M) \\ &= \sum_{\ell_1, \ell_2, \beta} \frac{\mathfrak{w}_1^{\beta \cap D_1} \dots \mathfrak{w}_{B_1}^{\beta \cap D_{B_1}} y_1^{\partial \beta \cap e_1} \dots y_n^{\partial \beta \cap e_n}}{(\ell_1 + \ell_2 + 1)!} \mathfrak{q}_{0, \ell_1 + \ell_2 + 1, \beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell_1} \mathbf{e}_i^M \mathfrak{b}_{\text{high}}^{\otimes \ell_2}, 1). \end{aligned} \quad (20.45)$$

It is easy to see that

$$\frac{\partial P_\beta}{\partial w_i} = \sum_{\ell_1, \ell_2} \frac{1}{(\ell_1 + \ell_2 + 1)!} \mathfrak{q}_{0, \ell_1 + \ell_2 + 1, \beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell_1} \mathbf{e}_i^M \mathfrak{b}_{\text{high}}^{\otimes \ell_2}, 1).$$

Therefore

$$i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(\mathbf{e}_i^M) = \frac{\partial \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}}{\partial w_i}(\mathfrak{y}), \quad (20.46)$$

as required.

(Case 3; $i = 1, \dots, B_1$): The equality (20.45) also holds in this case. Then, by [FOOO3] Lemma 9.2 we have

$$\begin{aligned} & i_{\text{qm}, (\mathfrak{b}(\mathfrak{y}), b(\mathfrak{y}))}^{T,*}(\mathbf{e}_i^M) \\ &= \sum_{\ell, \beta} (\beta \cap D_i) \frac{\mathfrak{w}_1^{\beta \cap D_1} \cdots \mathfrak{w}_{B_1}^{\beta \cap D_{B_1}} y_1^{\partial \beta \cap e_1} \cdots y_n^{\partial \beta \cap e_n}}{(\ell_1 + \ell_2 + 1)!} \mathfrak{q}_{0, \ell_1 + \ell_2 + 1, \beta}^T(\mathfrak{b}_{\text{high}}^{\otimes \ell}, 1). \end{aligned} \quad (20.47)$$

Using

$$\frac{\partial \mathfrak{w}_i^{\beta \cap D_i}}{\partial w_i} = (\beta \cap D_i) \mathfrak{w}_i^{\beta \cap D_i},$$

we obtain (20.46) also in this case. The proof of Theorem 20.23 is now complete. \square

21. SPECTRAL INVARIANTS AND QUASIMORPHISMS FOR TORIC MANIFOLDS

21.1. $\mu_e^{\mathfrak{b}}$ -heavyness of the Lagrangian fibers in toric manifolds. Let (M, ω) be a compact toric manifold, P its moment polytope. Let $\mathfrak{b} \in H^{\text{even}}(M; \Lambda_0)$. We consider the factorization

$$QH_{\mathfrak{b}}^*(M; \Lambda) \cong \prod_{\eta \in \text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})} QH_{\mathfrak{b}}(M; \eta)$$

corresponding to the one given in Proposition 20.22 via Theorem 20.18 so that $QH(M; \eta)$ is the factor corresponding to $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \eta)$.

Theorem 21.1. *Let $\eta = (\eta_1, \dots, \eta_n) \in \text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})$ and $e_{\eta} \in QH_{\mathfrak{b}}(M; \eta)$ be the corresponding idempotent. We put*

$$\mathbf{u}(\eta) = (\mathfrak{v}_T(\eta_1), \dots, \mathfrak{v}_T(\eta_n)) \in \text{Int } P.$$

Then the following holds:

- (1) $L(\mathbf{u}(\eta))$ is $\mu_{e_{\eta}}^{\mathfrak{b}}$ -heavy.
- (2) If η is a nondegenerate critical point, then $L(\mathbf{u}(\eta))$ is $\mu_{e_{\eta}}^{\mathfrak{b}}$ -superheavy.

Proof. Theorem 21.1 follows from Theorems 18.8, 20.23, Proposition 20.22 and the following lemma below. \square

Lemma 21.2. *Let $\mathfrak{k}\mathfrak{s}_{\mathfrak{b}}(e_{\eta}) = 1_{\eta} = [\mathfrak{P}]$ with $\mathfrak{P} \in \Lambda \langle \langle y, y^{-1} \rangle \rangle^{\mathring{P}}$. Then*

$$\mathfrak{P}(\eta) = 1.$$

Proof. The ring homomorphism

$$P \mapsto P(\eta) : \Lambda \langle \langle y, y^{-1} \rangle \rangle^{\mathring{P}} \rightarrow \Lambda$$

induces a ring homomorphism

$$\text{eval}_{\eta} : \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda) \rightarrow \Lambda.$$

The ring homomorphism eval_{η} is unital and so is surjective.

Let $\eta' \in \text{Crit}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}})$, $\eta' \neq \eta$ and $[P] \in \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \eta')$. By definition

$$(\widehat{y}_i - \eta'_i)^N [P] = 0$$

in $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \Lambda)$. Therefore applying eval_{η} we have

$$(\eta_i - \eta'_i)^N \text{eval}_{\eta}([P]) = 0.$$

Since $\eta_i - \eta'_i \neq 0$ for some i , we conclude $\text{eval}_{\eta}([P]) = 0$.

Therefore by Proposition 20.22, the homomorphism $\text{eval}_{\mathfrak{y}}$ is nonzero on the factor $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}; \mathfrak{y})$. Since $1_{\mathfrak{y}} = [\mathfrak{P}]$ is the unit of this factor, we conclude $\mathfrak{P}(\mathfrak{y}) = 1$, as required. \square

21.2. Calculation of the leading order term of the potential function in the toric case: review. We put

$$z_i = T^{\ell_i(\mathbf{0})} y_1^{\partial\beta_i \cap \mathbf{e}_1} \dots y_n^{\partial\beta_i \cap \mathbf{e}_n} \in \Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}. \quad (21.1)$$

We assume

$$\mathfrak{b} - \sum_{i=1}^{B_1} \bar{\mathfrak{b}}_i \mathbf{e}_i^M \in H^2(M; \Lambda_+) \oplus \bigoplus_{k \neq 1} H^{2k}(M; \Lambda_0), \quad (21.2)$$

where $\bar{\mathfrak{b}}_i \in \mathbb{C}$.

Theorem 21.3. *We have*

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}} = \mathfrak{b}_0 + \sum_{i=1}^m e^{\bar{\mathfrak{b}}_i} z_i + \sum_j T^{\lambda_j} P_j(z_1, \dots, z_m) \quad (21.3)$$

where $P_j \in \Lambda[z_1, \dots, z_m]$, $\lambda_j \in \mathbb{R}_{>0}$, $\lim_{j \rightarrow \infty} \lambda_j = \infty$.

In case (M, ω) is Fano and $\mathfrak{b} \in H^2(M; \Lambda_0)$, we have

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}} = \sum_{i=1}^m e^{w_i} z_i \quad (21.4)$$

where $\mathfrak{b} = \sum w_i \mathbf{e}_i^M$.

Proof. Theorem 21.3 is [FOOO3] Theorem 3.5. (See also [FOOO8] Theorem 8.2.) We sketch the proof below. We use the result of Cho-Oh [CO] for the proof. It is summarized in [FOOO2] Theorem 11.1 as follows.

- (1) If $\mathring{\mathcal{M}}_{1;0}(\beta) \neq \emptyset$, $\mu_L(\mathbf{u})(\beta) = 2$ then $\beta = \beta_j$ for $j = 1, \dots, m$, where β_j is as in (20.4). In this case $\mathring{\mathcal{M}}_{1;0}(\beta_j) = \mathcal{M}_{1;0}(\beta_j) = T^n$ and the evaluation map $\text{ev}_0^{\partial} : \mathcal{M}_{1;0}(\beta_j) \rightarrow L(\mathbf{u})$ has degree 1.
- (2) If $\mathcal{M}_{1;0}(\beta) \neq \emptyset$, $\beta \neq \beta_j$ ($j = 1, \dots, m$) then

$$\beta = \sum_{j=1}^m k_j \beta_j + \alpha,$$

where $\sum k_j > 0$, $k_j \geq 0$ and $\alpha \in \pi_2(M)$ with $\alpha \cap \omega > 0$.

Using this description we calculate terms of the right hand side of (20.43) as follows.

In case $\beta = \beta_j$ we have

$$\begin{aligned} & T^{\omega \cap \beta_j} \mathfrak{w}_1^{\beta_j \cap D_1} \dots \mathfrak{w}_{B_1}^{\beta_j \cap D_{B_1}} y_1^{\partial\beta_j \cap \mathbf{e}_1} \dots y_n^{\partial\beta_j \cap \mathbf{e}_n} P_{\beta_j}(w') \\ &= e^{w_j} z_j = (e^{\bar{\mathfrak{b}}_j} + (\text{higher order})) z_j. \end{aligned}$$

In case $\beta \neq \beta_j$ ($j = 1, \dots, m$) we have

$$T^{\omega \cap \beta} \mathfrak{w}_1^{\beta \cap D_1} \dots \mathfrak{w}_{B_1}^{\beta \cap D_{B_1}} y_1^{\partial\beta \cap \mathbf{e}_1} \dots y_n^{\partial\beta \cap \mathbf{e}_n} P_{\beta}(w') = T^{\alpha \cap \omega} \prod_{j=1}^m (e^{k_j w_j} z_j^{k_j}).$$

Therefore Theorem 21.3 follows. \square

21.3. Existence of Calabi quasimorphism on toric manifolds. In this subsection we complete the proof of Corollary 1.4. We begin with the following lemma

Lemma 21.4. *The set of vectors $(c_1, \dots, c_m) \in (\mathbb{C} \setminus \{0\})^m$ with the following properties is dense in $(\mathbb{C} \setminus \{0\})^m$:*

The function f defined by

$$f(y_1, \dots, y_n) = \sum_{i=1}^m c_i y_1^{\partial\beta \cap \mathbf{e}_1} \dots y_n^{\partial\beta \cap \mathbf{e}_n} \quad (21.5)$$

restricts to a Morse function on $(\mathbb{C} \setminus \{0\})^n$.

This lemma is proved in [Ku] (see [Iri1] Corollary 5.12, [FOOO2] Proposition 8.8 for the discussion in this context).

Corollary 21.5. *Write $\mathbf{b} = \sum_{i=1}^m \log c_i [D_i] \in H(M; \mathbb{C})$ and consider the sum*

$$\mathfrak{P}\mathfrak{D}_{\bar{\mathbf{b}},0} = \sum_{i=1}^m c_i z_i \in \Lambda[y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}].$$

Then the set of $(c_1, \dots, c_m) \in (\mathbb{C} \setminus \{0\})^m$ for which $\mathfrak{P}\mathfrak{D}_{\bar{\mathbf{b}},0}$ becomes a Morse function is dense in $(\mathbb{C} \setminus \{0\})^m$.

Proof. Suppose that $\mathfrak{P}\mathfrak{D}_{\bar{\mathbf{b}},0}$ is not a Morse function. Consider a degenerate critical point $\mathfrak{y} = (\mathfrak{y}_1, \dots, \mathfrak{y}_n)$ each of whose coordinates is a ‘formal Laurent power series’ of T . (We put ‘formal Laurent power series’ in the quote since the exponents of T are real numbers which are not necessarily integers.) By [FOOO2] Lemma 8.5, those series are convergent when we put $T = \epsilon$ for sufficiently small $\epsilon > 0$. Then for $c'_i = c_i \epsilon^{\ell_i(\mathbf{0})}$ the function (21.5) will not be a Morse function. Corollary 21.5 follows from this observation and Lemma 21.4. \square

Corollary 21.6. *For any compact toric manifold M there exists an element $\mathbf{b} \in H^{\text{even}}(M; \Lambda)$ such that $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$ is a Morse function.*

Proof. By [FOOO2] Theorem 10.4 we can prove that if $\mathfrak{P}\mathfrak{D}_{\bar{\mathbf{b}},0}$ is a Morse function then $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$ is also a Morse function. (Actually the case $\bar{\mathbf{b}} = 0$ is stated there. However the general case can be proved in the same way.) Therefore Corollary 21.6 follows from Corollary 21.5. \square

Corollary 1.4 follows immediately from Corollary 21.6, Proposition 20.22, Theorem 20.18 and Theorem 16.3. \square

21.4. Defect estimate of a quasimorphism $\mu_e^{\mathbf{b}}$. Using the calculations we have carried out, we can obtain some explicit estimates of the norm of the defect $\text{Def } \mu_e$ of spectral quasimorphism μ_e . We define

$$|\text{Def}|(\mu_e) = \sup_{\tilde{\psi}, \tilde{\phi}} |\mu_e(\tilde{\psi}\tilde{\phi}) - \mu_e(\tilde{\psi}) - \mu_e(\tilde{\phi})|.$$

We illustrate this estimate by an example.

We consider $(M, \omega) = \mathbb{C}P^n$ with moment polytope

$$\{(u_1, \dots, u_n) \mid u_i \geq 0, \sum u_i \leq 1\}.$$

Set $\mathbf{b} = \mathbf{0}$. It is well known that the small quantum cohomology $QH(\mathbb{C}P^n; \Lambda)$ is isomorphic to $\Lambda[x]/(x^{n+1} - T)$, where $x \in H^2(\mathbb{C}P^n; \mathbb{C})$ is the standard generator.

This is isomorphic to the direct product of $n + 1$ copies of Λ . Therefore we have $n + 1$ quasimorphisms μ_{e_k} . ($k = 0, \dots, n$.) It is actually defined on the Hamiltonian diffeomorphism group $\text{Ham}(\mathbb{C}P^n, \omega)$ itself. ([EP1] Section 4.3.) (It is unknown whether they are different from one another.)

Proposition 21.7. *Let e_k and μ_{e_k} for $k = 0, \dots, n$ be as above. Then*

$$|\text{Def}|(\mu_{e_k}) \leq \frac{12n}{n+1}.$$

Proof. We have

$$\mathfrak{P}\mathfrak{D}_0 = y_1 + \dots + y_n + T(y_1 \dots y_n)^{-1}.$$

See for example [FOOO2] Example 5.2.

Let $\chi_k = \exp(2\pi k \sqrt{-1}/(n+1))$. The critical points of $\mathfrak{P}\mathfrak{D}_0$ are

$$\mathfrak{y}_k = T^{1/(n+1)}(\chi_k, \dots, \chi_k), \quad k = 0, 1, \dots, n.$$

We put

$$P_k = \frac{\prod_{i \neq k} (y_1 - T^{1/(n+1)} \chi_i)}{\prod_{i \neq k} (T^{1/(n+1)} \chi_k - T^{1/(n+1)} \chi_i)}.$$

Since

$$P_k(\mathfrak{y}_\ell) = \begin{cases} 0 & k \neq \ell, \\ 1 & k = \ell, \end{cases}$$

it follows that $[P_k] = \mathfrak{t}\mathfrak{s}_0(e_{\mathfrak{y}_k})$ in the Jacobian ring. Therefore, using $\mathfrak{t}\mathfrak{s}_0(x) = [y_1]$ also, we have

$$\mathfrak{v}_q(e_{\mathfrak{y}_k}) = -\mathfrak{v}_T(e_{\mathfrak{y}_k}) = \frac{n}{n+1}.$$

Proposition 21.7 now follows from Remark 16.8 (1). \square

Note we chose our symplectic form ω so that $\int_{\mathbb{C}P^1} \omega = 1$. (See (20.5) and Remark 20.3.)

22. LAGRANGIAN TORI IN k -POINTS BLOW UP OF $\mathbb{C}P^2$ ($k \geq 2$).

In this section, we prove Theorem 1.11 (3) in the case of k -points blow up of $\mathbb{C}P^2$ ($k \geq 2$). We use the example of [FOOO3] Section 5, which we review now.

We first consider 2-points blow up M of $\mathbb{C}P^2$. We put a toric Kähler form on it $\omega_{\alpha, \beta}$ such that the moment polytope is given by

$$P_{\alpha, \beta} = \{(u_1, u_2) \mid 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1 - \alpha, \beta \leq u_1 + u_2 \leq 1\}. \quad (22.1)$$

Here

$$(\alpha, \beta) \in \{(\alpha, \beta) \mid 0 \leq \alpha, \beta, \alpha + \beta \leq 1\}. \quad (22.2)$$

We are interested in the case $\beta = (1 - \alpha)/2$ and write $M_\alpha = (M, \omega_{\alpha, (1-\alpha)/2})$ where $\alpha > 1/3$. We denote

$$D_1 = \pi^{-1}(\partial P_{\alpha, (1-\alpha)/2} \cap \{(u_1, u_2) \mid u_2 = 0\})$$

and put

$$\mathfrak{b}_\kappa = T^\kappa PD([D_1]) \in H^2(M_\alpha; \Lambda_+), \quad \kappa > 0. \quad (22.3)$$

Then by (21.4) we have

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}_\kappa} = y_1 + e^{T^\kappa} y_2 + T^{1-\alpha} y_2^{-1} + T y_1^{-1} y_2^{-1} + T^{-(1-\alpha)/2} y_1 y_2. \quad (22.4)$$

Now consider a family of Lagrangian torus fibers

$$L(u) = L(u, (1 - \alpha)/2), \quad (22.5)$$

for $(1 - \alpha)/2 < u < (1 + \alpha)/4$. Then for any such u we can show the following. Note that $\alpha > 1/3$ implies $(1 - \alpha)/2 < 1/3 < (1 + \alpha)/4$.

Theorem 22.1. *If $1/3 \leq u < (1 + \alpha)/4$, we take $\kappa(u) = (1 + \alpha)/2 - 2u > 0$. If $(1 - \alpha)/2 < u < 1/3$, we take $\kappa(u) = u - (1 - \alpha)/2 > 0$. Then $L(u) \subset M_\alpha$ is $\mu_e^{\mathfrak{b}_{\kappa(u)}}$ -superheavy with respect to an appropriate idempotent e of $QH_{\mathfrak{b}_{\kappa(u)}}^*(M_\alpha; \Lambda)$.*

Proof. Let $\mathbf{u} = (u, (1 - \alpha)/2)$. We put

$$y_1^{\mathbf{u}} = T^{-u_1} y_1 = T^{-u} y_1, \quad y_2^{\mathbf{u}} = T^{-u_2} y_2 = T^{-(1-\alpha)/2} y_2$$

in (22.4) to obtain

$$\begin{aligned} \mathfrak{P}\mathfrak{D}_{\mathfrak{b}_{\kappa(u)}} = & T^u y_1^{\mathbf{u}} + e^{T^{\kappa(u)}} T^{(1-\alpha)/2} y_2^{\mathbf{u}} + T^{(1-\alpha)/2} (y_2^{\mathbf{u}})^{-1} \\ & + T^{(1+\alpha)/2-u} (y_1^{\mathbf{u}})^{-1} (y_2^{\mathbf{u}})^{-1} + T^u y_1^{\mathbf{u}} y_2^{\mathbf{u}}. \end{aligned} \quad (22.6)$$

See [FOOO3] (5.10). We first consider the case that $1/3 < u < (1 + \alpha)/4$ and $\kappa(u) = (1 + \alpha)/2 - 2u$. Then the calculation in Case 1 of [FOOO3] Section 5 shows that the potential function $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}_{\kappa(u)}}$ has nondegenerate critical points $\mathfrak{y}(u) = (\mathfrak{y}_1(u), \mathfrak{y}_2(u))$ such that

$$(T^{-u} \mathfrak{y}_1(u), T^{-(1-\alpha)/2} \mathfrak{y}_2(u)) \equiv (\pm\sqrt{-2}, -1) \pmod{\Lambda_+}.$$

Each of them corresponds to an idempotent $e_{\mathfrak{y}(u)}$ of $QH_{\mathfrak{b}_{\kappa(u)}}(M_\alpha; \Lambda)$. Theorem 21.1 implies that $L(u)$ is $\mu_{e_{\mathfrak{y}(u)}}^{\mathfrak{b}_{\kappa(u)}}$ -superheavy. When $u = 1/3$ and $\kappa(u) = (1 + \alpha)/2 - 2u$, Case 4 of [FOOO3] Section 5 shows that there are nondegenerate critical points. (Note that we are using \mathfrak{b}_κ as (22.3) so $w = 1$ in [FOOO3] (5.14).) If $(1 - \alpha)/2 < u < 1/3$ and $\kappa(u) = u - (1 - \alpha)/2$, Case 3 of [FOOO3] Section 5 shows that there is a nondegenerate critical point as well. Thus Theorem 22.1 follows from Theorem 21.1. \square

Proof of Theorem 1.11 (3). Since $\mathfrak{y}(u)$ is a nondegenerate critical point, Theorem 20.18 and Proposition 20.22 imply that $e = e_{\mathfrak{y}(u)}$ is the unit of the direct factor of $QH_{\mathfrak{b}_{\kappa(u)}}(M_\alpha; \Lambda)$ that is isomorphic to Λ . Therefore by Theorem 16.3 $\mu_{e_{\mathfrak{y}(u)}}^{\mathfrak{b}_{\kappa(u)}}$ is a Calabi quasimorphism. By Corollary 1.10, the set

$$\left\{ \mu_{e_{\mathfrak{y}(u)}}^{\mathfrak{b}_{\kappa(u)}} \right\}_{u \in ((1-\alpha)/2, (1+\alpha)/4)}$$

is linearly independent. Thus we have constructed a continuum of linearly independent Calabi quasimorphisms parametrized by $u \in ((1 - \alpha)/2, (1 + \alpha)/4)$. The proof of Theorem 1.11 in case of two points blow up of $\mathbb{C}P^2$ is complete.

To prove the existence of a continuum of linearly independent Calabi quasimorphisms in case of three points blow up of $\mathbb{C}P^2$, we consider the Kähler toric surface (M, ω) whose moment polytope is

$$P_{\alpha, (1-\alpha)/2} \setminus \{(u_1, u_2) \mid 1 - \epsilon < u_2\}$$

for sufficiently small ϵ . Then (M, ω) is a three points blow up of $\mathbb{C}P^2$. Its potential function is

$$(22.6) + T^{1-\epsilon} y_1^{-1}.$$

It is easy to see that the extra term $T^{1-\epsilon} y_1^{-1}$ is of higher order, when $(\mathfrak{v}_T(y_1), \mathfrak{v}_T(y_2)) = (u, (1 - \alpha)/2)$, $u \in (1/3, (1 + \alpha)/4)$. So by the same argument as the case of two

points blow up, we can prove Theorem 1.11. For $k > 3$ points blow up, we can repeat the same argument. (See [FOOO2] page 111.) \square

23. LAGRANGIAN TORI IN $S^2 \times S^2$

In this section we prove Theorem 1.11 in the case of $S^2 \times S^2$, which is equipped with the symplectic structure $\omega \oplus \omega$. We also prove Theorem 1.13. We first recall the description of the family of Lagrangian tori constructed in [FOOO5].

23.1. Review of the construction from [FOOO5]. We consider the toric Hirzebruch surface $F_2(\alpha)$ ($\alpha > 0$) whose moment polytope is

$$P(\alpha) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_i \geq 0, u_2 \leq 1 - \alpha, u_1 + 2u_2 \leq 2\}. \quad (23.1)$$

Note that $F_2(\alpha)$ is not Fano but nef, i.e. every holomorphic sphere has non-negative Chern number. In fact, the divisor $D_1 \cong \mathbb{CP}^1$ associated to the facet $\partial_1 P(\alpha) = \{u \in \partial P(\alpha) \mid u_2 = 1 - \alpha\}$ has $c_1(D_1) = 0$.

Theorem 23.1 (Theorem 2.2 [FOOO5]). *We put $\mathbf{b} = \mathbf{0}$. The potential function $\mathfrak{P}\mathfrak{D}_0$ of $F_2(\alpha)$ has the form*

$$\mathfrak{P}\mathfrak{D}_0 = y_1 + y_2 + T^2 y_1^{-1} y_2^{-2} + T^{1-\alpha} (1 + T^\alpha) y_2^{-1}. \quad (23.2)$$

We consider the limit $\alpha \rightarrow 0$ of the Hirzebruch surface $F_2(\alpha)$. At $\alpha = 0$, the limit polytope is the triangle

$$P(0) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_i \geq 0, u_2 \leq 1, u_1 + 2u_2 \leq 2\} \quad (23.3)$$

and the limit $F_2(0)$ is an orbifold with a singularity of the form $\mathbb{C}^2/\{\pm 1\}$. We cut out a neighborhood of the singularity of $F_2(0)$ and paste the Milnor fiber back into the neighborhood to obtain the desired manifold. We denote it by $\widehat{F}_2(0)$.

Consider the preimage $Y(\varepsilon)$ of $P(\varepsilon) \subset P(0)$, $0 < \varepsilon < 1$, under the moment map $\pi : F_2(0) \setminus \{O\} \rightarrow P(0) \setminus \{(0, 1)\}$, where O is the singularity of $F_2(0)$. We can put a natural glued symplectic form on $\widehat{F}_2(0) = Y(\varepsilon) \cup D_r(T^*S^2)$ in a way that the given toric symplectic form on $Y(\varepsilon)$ is unchanged on $Y(\varepsilon) \setminus N(\varepsilon) \subset Y(\varepsilon) \setminus \partial Y(\varepsilon)$, where $N(\varepsilon)$ is a collar neighborhood of $\partial Y(\varepsilon)$. Since $H^2(S^3/\{\pm 1\}; \mathbb{Q}) = 0$, the glued symplectic form does not depend on the choices of $\varepsilon > 0$ or the gluing data up to the symplectic diffeomorphism. This symplectic manifold is symplectomorphic to $(S^2, \omega_{\text{std}}) \times (S^2, \omega_{\text{std}})$ (Proposition 5.1 [FOOO5].) In other words, we have symplectomorphisms

$$\phi_\varepsilon : (\widehat{F}_2(0), \omega_\varepsilon) \rightarrow (S^2 \times S^2, \omega_{\text{std}} \oplus \omega_{\text{std}}). \quad (23.4)$$

We denote

$$T(\rho) = \phi_\varepsilon(L(1/2 - \rho, 1/2 + \rho)), \quad 0 \leq \rho < \frac{1}{2} \quad (23.5)$$

where $L(1/2 - \rho, 1/2 + \rho) = \pi^{-1}(1/2 - \rho, 1/2 + \rho)$ regarded as a Lagrangian submanifold of $(\widehat{F}_2(0), \omega_\varepsilon)$. We refer to Sections 3 and 4 of [FOOO5] for the detailed explanation of this construction.

23.2. Superheavyness of $T(\rho)$. Recall from Section 5 [FOOO5] that we have a family $\bigcup_{a \in \mathbb{C}} X_a$ where X_a is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ for $a \neq 0$ and X_0 is biholomorphic to F_2 . (See Lemma 5.1 [FOOO5].) The smooth trivialization of the simultaneous resolution $\bigcup_{a \in \mathbb{C}} X_a$ of $F_2(0)$ constructed in Section 6 [FOOO5] identifies the homology class $[D_1]$ in X_0 and $[S_{\text{van}}^2]$ in X_a . Beside this, the relative homology class β_1 in X_0 which satisfies $\beta_1 \cap D_1 = 1$ and does not intersect with other toric divisors can be also regarded as a homology class in X_a . The homology classes β_1 and $\beta_1 + [S_{\text{van}}^2]$ satisfy the relations

$$\beta_1 \cap [S_{\text{van}}^2] = 1, \quad (\beta_1 + [S_{\text{van}}^2]) \cap [S_{\text{van}}^2] = -1. \quad (23.6)$$

We consider the cohomology class

$$\mathbf{b}(\rho) = T^\rho PD[S_{\text{van}}^2] \in H^2(\widehat{F}_2(0), \Lambda_+).$$

Using the 4-dimensionality and special properties of $\widehat{F}_2(0)$, we proved the following in [FOOO5] Theorem 8.2.

Lemma 23.2.

$$H^1(T(u); \Lambda_0) \subset \{b \in H^{\text{odd}}(T(u); \Lambda_0) \mid (\mathbf{b}(\rho), b) \in \widehat{\mathcal{M}}_{\text{def, weak}}(T(u); \Lambda_0)\}.$$

In [FOOO5], we showed that the potential function for $T(0)$, i.e., $\mathbf{u}_0 = (1/2, 1/2)$ is

$$\mathfrak{P}\mathfrak{D} = T^{1/2}(y_1^{\mathbf{u}_0} + y_2^{\mathbf{u}_0} + (y_1^{\mathbf{u}_0})^{-1}(y_2^{\mathbf{u}_0})^{-2} + 2(y_2^{\mathbf{u}_0})^{-1}).$$

We find that there are two critical points $\pm(1/2, 2)$, see [FOOO5] Digression 4.1. Hence there exist two $b \in H^1(T(0); \Lambda_0)$ modulo $H^1(T(0); 2\pi\sqrt{-1}\mathbb{Z})$ such that $HF((T(0), b); \Lambda) \neq 0$.

When we consider the bulk deformation by $\mathbf{b}(\rho)$, (23.6) and Theorem 23.1 imply that the potential function of $T(\mathbf{u})$ with bulk, $\mathfrak{P}\mathfrak{D}_{\mathbf{b}(\rho)}$, becomes

$$\begin{aligned} \mathfrak{P}\mathfrak{D}_{\mathbf{b}(\rho)} &= T^{u_1}y_1^{\mathbf{u}} + T^{u_2}y_2^{\mathbf{u}} + T^{2-u_1-2u_2}(y_1^{\mathbf{u}})^{-1}(y_2^{\mathbf{u}})^{-2} \\ &\quad + (e^{T^\rho} + e^{-T^\rho})T^{1-u_2}(y_2^{\mathbf{u}})^{-1}. \end{aligned} \quad (23.7)$$

(See [FOOO3] Theorem 3.5 and [FOOO5] Formula (47).) Now we put

$$2\rho = u_2 - u_1 = u_2 - (1 - u_2) = 2u_2 - 1$$

and consider (23.7) at $\mathbf{u} = (u_1, u_2)$ for some ρ . Namely, $\mathbf{u} = (1/2 - \rho, 1/2 + \rho)$. Then the potential function with bulk $\mathbf{b}(\rho)$ of $T(0)$ is written as

$$T^{1/2}(y_1^{\mathbf{u}_0} + y_2^{\mathbf{u}_0} + (y_1^{\mathbf{u}_0})^{-1}(y_2^{\mathbf{u}_0})^{-2} + (e^{T^\rho} + e^{-T^\rho})(y_2^{\mathbf{u}_0})^{-1}).$$

See Formula (47) in [FOOO5] with $u_1 = u_2 = 1/2$. There are two critical points, which are $(\eta_1^0(\rho), \eta_2^0(\rho)) = (\epsilon(e^{T^{\rho/2}} + e^{-T^{\rho/2}})^{-1}, \epsilon(e^{T^{\rho/2}} + e^{-T^{\rho/2}}))$ with $\epsilon = \pm 1$. Hence $b^0(\rho) = b(\eta^0(\rho)) = (\log \eta_1^0(\rho), \log \eta_2^0(\rho)) \in H^1(T(0); \Lambda_0)$,

$$HF((T(0), (\mathbf{b}(\rho), b^0(\rho))); \Lambda) \neq 0.$$

For $T(\rho)$, the potential function with bulk \mathbf{b}_ρ is written as

$$T^{1/2-\rho}(y_1^{\mathbf{u}} + T^{2\rho}y_2^{\mathbf{u}} + (y_1^{\mathbf{u}})^{-1}(y_2^{\mathbf{u}})^{-2} + (e^{T^\rho} + e^{-T^\rho})(y_2^{\mathbf{u}})^{-1}).$$

See Formula (47) in [FOOO5] with $u_1 = 1/2 - \rho, u_2 = 1/2 + \rho$. There are two critical points, which are $(\eta_1(\rho), \eta_2(\rho)) = (\epsilon T^\rho(e^{T^{\rho/2}} - e^{-T^{\rho/2}})^{-1}, -\epsilon T^{-\rho}(e^{T^{\rho/2}} - e^{-T^{\rho/2}}))$. It follows that for $b(\rho) = b(\eta(\rho)) = (\log \eta_1(\rho), \log \eta_2(\rho)) \in H^1(T(\rho); \Lambda_0)$,

$$HF((T(\rho), (\mathbf{b}(\rho), b(\rho))); \Lambda) \neq 0.$$

In sum, we have

Lemma 23.3. (1) *There exist two $b \in H^1(T(0); \Lambda_0)/H^1(T(0); 2\pi\sqrt{-1}\mathbb{Z})$ such that*

$$HF((T(0), b); \Lambda) \neq 0.$$

(2) *There exist two $b^0(\rho) \in H^1(T(0); \Lambda_0)/H^1(T(0); 2\pi\sqrt{-1}\mathbb{Z})$ such that*

$$HF((T(0), (\mathbf{b}(\rho), b^0(\rho)); \Lambda) \neq 0.$$

(3) *There exist two $b(\rho) \in H^1(T(\rho); \Lambda_0)/H^1(T(\rho); 2\pi\sqrt{-1}\mathbb{Z})$ such that*

$$HF((T(\rho), (\mathbf{b}(\rho), b(\rho)); \Lambda) \neq 0.$$

Using this, we show the following:

Theorem 23.4. (1) *There exists an idempotent e of a field factor of $QH(S^2 \times S^2; \Lambda)$ such that $T(0)$ is μ_e -superheavy.*

(2) *For any $0 \leq \rho < \frac{1}{2}$, there exist idempotents e_ρ and e_ρ^0 , each of which is an idempotent of a field factor of $QH_{\mathbf{b}(\rho)}(S^2 \times S^2; \Lambda)$ such that $T(\rho)$ is $\mu_{e_\rho}^{\mathbf{b}(\rho)}$ -superheavy and $T(0)$ is $\mu_{e_\rho^0}^{\mathbf{b}(\rho)}$ -superheavy.*

Proof. We observe that $QH(S^2 \times S^2; \Lambda)$ and $QH_{\mathbf{b}(\rho)}(S^2 \times S^2; \Lambda)$ are semi-simple. For this purpose, we consider the toric structure as the monotone product of S^2 . Let $\mathbf{b} = a[S_{\text{van}}^2]$, $a \in \Lambda_+$. ($\mathbf{b} = \mathbf{b}(\rho)$ if $a = T^\rho$, while $\mathbf{b} = 0$ if $a = 0$.) We pick points pt_1 , resp. pt_2 , on the first, resp. second, factor of $S^2 \times S^2$ in the hemisphere in the class β_1, β_2 , which contribute to the coefficients of y_1, y_2 in the potential function, respectively. We fix a representative $[S^2 \times pt_2] - [pt_1 \times S^2]$ of the homology class $[S_{\text{van}}^2]$. The potential function of $S_{\text{eq}}^1 \times S_{\text{eq}}^1$ with bulk \mathbf{b} is written as

$$\mathfrak{P}\mathfrak{D}_{\mathbf{b}} = T^{1/2}(e^a y_1 + y_1^{-1} + e^{-a} y_2 + y_2^{-1}).$$

It has four nondegenerate critical points $(\epsilon_1 e^{-a/2}, \epsilon_2 e^{a/2})$ with $\epsilon_1, \epsilon_2 = \pm 1$. The critical values are $2(\epsilon_1 e^{a/2} + \epsilon_2 e^{-a/2})T^{1/2}$. By Theorem 6.1 in [FOOO2] (Fano toric case) and Theorem 1.1 in [FOOO6], we find that the quantum cohomology with bulk deformation by \mathbf{b} is factorized into four copies of Λ :

$$QH_{\mathbf{b}}(S^2 \times S^2; \Lambda) \cong \bigoplus_{i=1}^4 \Lambda \mathbf{e}_i^{\mathbf{b}}.$$

Here $\mathbf{e}_1^{\mathbf{b}}, \dots, \mathbf{e}_4^{\mathbf{b}}$ are the idempotents corresponding to the critical points of $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$ with $(\epsilon_1, \epsilon_2) = (1, 1), (1, -1), (-1, 1), (-1, -1)$, respectively. (When $\mathbf{b} = 0$, we simply write them as \mathbf{e}_i .) In particular, it is semi-simple.

By Lemma 23.3 (1) and (3), there exists b , resp. $b(\rho)$, such that $HF((T(0), b); \Lambda) \neq 0$, resp. $HF((T(\rho), (\mathbf{b}(\rho), b(\rho)); \Lambda) \neq 0$. Hence Theorem 3.8.62 in [FOOO1] with taking (3.8.36.2) in Theorem 3.8.32 into account implies that

$$i_{\text{qm}, T(0), b}^* : QH(S^2 \times S^2; \Lambda) \rightarrow HF((T(0), b); \Lambda),$$

$$i_{\text{qm}, T(\rho), (\mathbf{b}(\rho), b(\rho))}^* : QH_{\mathbf{b}(\rho)}(S^2 \times S^2; \Lambda) \rightarrow HF((T(\rho), (\mathbf{b}(\rho), b(\rho)); \Lambda).$$

send the unit to the unit. In particular, there is at least one idempotent $e_0 \in QH(S^2 \times S^2; \Lambda)$, resp. $e_\rho \in QH_{\mathbf{b}(\rho)}(S^2 \times S^2; \Lambda)$ such that $i_{\text{qm}, T(0), b_i}^*(e_0) \neq 0$, resp. $i_{\text{qm}, T(\rho), (\mathbf{b}(\rho), b(\rho))}^*(e_\rho) \neq 0$. Hence $T(0)$ is μ_e -superheavy and $T(\rho)$ is $\mu_{e_\rho}^{\mathbf{b}(\rho)}$ -superheavy. \square

The sphere S_{van}^2 is a Lagrangian submanifold, which is disjoint from $T(\rho)$. We have the following

Lemma 23.5. *The Lagrangian sphere S_{van}^2 , which is the anti-diagonal in $S^2 \times S^2$, is unobstructed and*

$$HF(S_{\text{van}}^2; \Lambda) \cong H(S_{\text{van}}^2; \Lambda) \neq 0.$$

Proof. Note that the anti-diagonal in $S^2 \times S^2$ can be seen as a fixed point set of an anti-symplectic involution. Then Theorem 1.3 with $k = 0, \ell = 0$ in [FOOO4] implies that $\mathfrak{m}_0(1) = 0$, since the Maslov index of any holomorphic disc in $(S^2 \times S^2, S_{\text{van}}^2)$ is divisible by 4. See also [FOOO4] Corollary 1.6. The second assertion follows from [FOOO1] Theorem D (D.3). \square

Lemma 23.5 implies that the value of the potential function is zero and there is only one bounding cochain $0 \in H^1(S_{\text{van}}^2; \Lambda_0)$ up to gauge equivalence in this case. By the same argument as in the case of $T(0)$, we find an idempotent e' of a field factor of $QH(S^2 \times S^2; \Lambda)$ and such that $i_{\text{qm}, S_{\text{van}}^2}^*(e') \neq 0$.

Since each of e_0, e' is an idempotent of a field factor of $QH(S^2 \times S^2; \Lambda)$ and e_ρ is an idempotent of a field factor of $QH_{\mathfrak{b}(\rho)}(S^2 \times S^2; \Lambda)$, there exist corresponding Calabi quasimorphisms $\mu_{e_0}^0, \mu_{e'}, \mu_{e_\rho}^{\mathfrak{b}(\rho)}$ from $\widetilde{\text{Ham}}(S^2 \times S^2)$ to \mathbb{R} . Since $T(\rho)$, $\rho \in [0, 1/2)$ and S_{van}^2 are mutually disjoint, Corollary 1.10 implies Theorem 1.11 (2). This completes the proof of Theorem 1.11 (2). \square

Furthermore, since homogenous quasimorphisms are homomorphisms on abelian subgroups and $\pi_1(\text{Ham}(S^2 \times S^2)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ [Gr], they descend to quasimorphisms on $\text{Ham}(S^2 \times S^2)$. We denote them by $\bar{\mu}_{e_0}^0, \bar{\mu}_{e'},$ and $\bar{\mu}_{e_\rho}^{\mathfrak{b}(\rho)}$. Thus we also obtain the following.

Corollary 23.6. *We have linearly independent Calabi quasimorphisms $\bar{\mu}_{e_0}^0, \bar{\mu}_{e'}$ and $\bar{\mu}_{e_\rho}^{\mathfrak{b}(\rho)}$ from $\text{Ham}(S^2 \times S^2)$.*

Remark 23.7. Generally, let (M, ω) be a closed symplectic manifold. Suppose that $\widetilde{\text{Ham}}(M, \omega)$ has infinitely many linearly independent homogeneous quasimorphisms μ_i . We give a sufficient condition for the existence of infinitely many linearly independent homogeneous quasimorphisms on $\text{Ham}(M, \omega)$. Suppose that $\pi_1(\text{Ham}(M, \omega))$ is finitely generated. Pick generators $\tilde{\phi}_1, \dots, \tilde{\phi}_A \in \widetilde{\text{Ham}}(M, \omega)$.

Proposition 23.8. *Under the assumption above, if there are infinitely many linearly independent homogeneous quasimorphisms μ_i on $\widetilde{\text{Ham}}(M, \omega)$, then there are infinitely many linearly independent homogeneous quasimorphisms on $\text{Ham}(M, \omega)$. The same statement holds for Calabi quasimorphisms.*

Proof. Let K be a maximal integer such that, for some i_1, \dots, i_K ,

$$(\mu_{i_j}(\tilde{\phi}_1), \dots, \mu_{i_j}(\tilde{\phi}_A)) \in \mathbb{R}^A, \quad j = 1, \dots, K$$

are linearly independent. We arrange the ordering such that $i_1 = 1, \dots, i_K = K$. For $k > K$, we can find $a_i(k) \in \mathbb{R}$ such that

$$\mu'_k(\tilde{\phi}_i) = \mu_k(\tilde{\phi}_i) - \sum_{i=1}^K a_i(k) \mu_i(\tilde{\phi}_i)$$

are zero for $i = 1, \dots, A$. Since the restriction of a homogeneous quasimorphism on an abelian subgroup is a homomorphism, μ'_k vanishes on $\pi_1(\text{Ham}(S^2 \times S^2))$,

which we regard as a subgroup of $\widetilde{\text{Ham}}(M, \omega)$. Therefore μ'_k , $k > K$, descends to a homogeneous quasimorphism on $\text{Ham}(M, \omega)$. Linear independence for μ'_k , $k > K$, follows from the one for μ_i .

For the statement concerning Calabi quasimorphisms, we take one more μ_{K+1} . Then, for $k > K + 1$, choose $a_i(k)$, $i = 1, \dots, K + 1$ such that $\mu'_k(\tilde{\phi}_i) = \mu_k(\tilde{\phi}_i) - \sum_{i=1}^{K+1} a_i(k) \mu_i(\tilde{\phi}_i)$ are zero for $i = 1, \dots, A$ and $\sum_{i=1}^{K+1} a_i(k) \neq 1$. Then after a suitable rescaling, μ'_k becomes a Calabi quasimorphism. \square

Remark 23.9. In case M is either a k (≥ 3) points blow up of $\mathbb{C}P^2$ or cubic surface, we can descend our family of Calabi quasimorphisms on $\widetilde{\text{Ham}}(M, \omega)$ to one on $\text{Ham}(M, \omega)$ in the same way as above if we can show that $\pi_1(\text{Ham}(M, \omega))$ is a finitely generated group.

Next, we prove Theorem 1.13 which follows from Theorem 23.4 together with Theorem 18.7.

Proof of Theorem 1.13. Note that $T(u)$ in Theorem 1.13 is $T(\rho)$. Since $S_{\text{eq}}^1 \times S_{\text{eq}}^1$ is the unique Lagrangian torus fiber with respect to the monotone toric structure on $S^2 \times S^2$, $S_{\text{eq}}^1 \times S_{\text{eq}}^1$ is superheavy with respect to the quasimorphism μ_e^b associated with any idempotent e of the field factor of $QH_{\mathfrak{b}_\rho}(S^2 \times S^2; \Lambda)$. (Here we consider $\mathfrak{b} = \mathfrak{b}_\rho = T^\rho[S_{\text{van}}^2]$.)

From Theorem 23.4 we know that $T(\rho)$ is superheavy with respect to the quasimorphism $\mu_{e_{\eta(\rho)}}^{b(\rho)}$ associated with a suitable idempotent $e_{\eta(\rho)}$ of $QH_{\mathfrak{b}_\rho}(S^2 \times S^2; \Lambda)$. Since the superheaviness is invariant under symplectomorphisms, $\varphi(T(\rho))$ is also $\mu_{e_{\eta(\rho)}}^{b(\rho)}$ -superheavy for any $\varphi \in \text{Ham}(S^2 \times S^2)$. Since superheavy sets with respect to the same quasimorphism must intersect by Theorem 18.7, we have

$$\varphi(T(\rho)) \cap (S_{\text{eq}}^1 \times S_{\text{eq}}^1) \neq \emptyset.$$

\square

Remark 23.10. We can also prove Theorem 23.4 in a way similar to the toric case as follows. One can see a similar argument in Section 24. For $\alpha > 0$ the map

$$\mathfrak{ts}_{\mathfrak{b}(\rho)} : QH_{\mathfrak{b}(\rho)}(F_2(\alpha); \Lambda) \rightarrow \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda)$$

is a ring homomorphism by [FOOO6] Theorem 9.1. (Here we use $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}$ of the toric manifold $F_2(\alpha)$.) Since we can take limit $\alpha \rightarrow 0$, we find that

$$\mathfrak{ts}_{\mathfrak{b}(\rho)} : QH_{\mathfrak{b}(\rho)}(\widehat{F}_2(0); \Lambda) \rightarrow \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda)$$

is also a ring homomorphism. (Here $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}$ is as in (23.7).)

Lemma 23.11. $\mathfrak{ts}_{\mathfrak{b}(\rho)}$ is a surjective.

Proof. We can check that $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}$ has exactly 4 critical points if $\rho \in (0, 1)$ and has exactly 2 critical points in case $\rho = 0$. We can also check that those critical points are nondegenerate. Therefore $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda) \cong \Lambda^4$ if $\rho \in (0, 1)$ and $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda) \cong \Lambda^2$ if $\rho = 0$. We put

$$z_1 = T^{1-u_2}(y_2^{\mathbf{u}})^{-1}, \quad z_2 = T^{u_1}y_1^{\mathbf{u}}, \quad z_3 = T^{u_2}y_2^{\mathbf{u}}, \quad z_4 = T^{2-u_1-2u_2}(y_1^{\mathbf{u}})^{-1}(y_2^{\mathbf{u}})^{-2}.$$

In the same way as [FOOO6] Lemma 2.3, we can show that z_1, \dots, z_4 generate a Λ -subalgebra that is dense in $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$. (See Definition 20.12 for the notation.)

Therefore, since $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda)$ is finite dimensional, they generate $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda)$ as Λ -algebra.

Let D_i ($i = 1, \dots, 4$) be the divisors of X associated to the facets $u_2 = 1 - \alpha$, $u_1 = 0$, $u_2 = 0$, $u_1 + 2u_2 = 2$ respectively. It is easy to see that

$$\mathfrak{ts}_{\mathfrak{b}(\rho)}(PD[D_i]) = z_i$$

for $i = 2, 3, 4$ and

$$\mathfrak{ts}_{\mathfrak{b}(\rho)}(PD[D_1]) = (e^{T^\rho} - e^{-T^\rho})z_1.$$

The lemma follows. \square

Let $\rho \neq 0$. Then $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}; \Lambda) \cong \Lambda^4$. Since the Betti number of $\widehat{F}_0(0)$ is 4, Lemma 23.11 implies that $\mathfrak{ts}_{\mathfrak{b}(\rho)}$ is an isomorphism.

Let $\rho = 0$. Using the fact that $QH_{\mathfrak{b}(0)}(X; \Lambda)$ is semisimple, and $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(0)}; \Lambda) \cong \Lambda^2$, we can find $e_{\mathfrak{b}(0)}$ that is a unit of the direct factor $\cong \Lambda$ of $QH_{\mathfrak{b}(0)}(X; \Lambda)$, such that $\mathfrak{ts}_{\mathfrak{b}(0)}(e_{\mathfrak{b}(0)}) \neq 0$. (In the case $\rho \neq 0$, existence of such $e_{\mathfrak{b}(\rho)}$ is immediate from the fact that $\mathfrak{ts}_{\mathfrak{b}(\rho)}$ is an isomorphism.)

Thus in a way similar to the proof of Theorem 20.23 and Lemma 21.2 we find that

$$i_{\text{qm}, (\mathfrak{b}(\rho), \mathfrak{b}(\eta^u))}^*(e_{\mathfrak{b}(\rho)}) \neq 0.$$

In fact, we can use a de Rham representative of the Poincaré dual to $[S_{\text{van}}^2]$ that is supported in a neighborhood of S_{van}^2 and in particular disjoint from $T(\rho)$. Therefore the above calculation of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}$ makes sense in the homology level.

23.3. Critical values and eigenvalues of $c_1(M)$. In this subsection we see the relation between the idempotent e_ρ and the critical points of the potential function of $T(\rho)$ with the bulk deformation $\mathfrak{b}(\rho)$ if $\rho \in (0, 1/2)$. We give a digression on the critical values of the potential function and the eigenvalues of the quantum multiplication by the first Chern class $c_1(M)$. We start with an easy observation.

Lemma 23.12. *For an oriented Lagrangian submanifold $L \subset M$, there is a cycle D of codimension 2 in $M \setminus L$ such that the Maslov index is equal to twice of the intersection number with D , i.e., $\mu(\beta) = 2\beta \cdot D$ for any $\beta \in H_2(M, L; \mathbb{Z})$.*

Proof. Since L is an oriented Lagrangian submanifold, the top exterior power $\bigwedge_{\mathbb{C}}^n TM$ is a trivial complex line bundle, where $2n = \dim M$. Moreover, the volume form of L gives a non-vanishing section s_L of $\bigwedge_{\mathbb{C}}^n TM|_L$. We extend s_L to a section s of $\bigwedge_{\mathbb{C}}^n TM$, which is transversal to the zero section. Then the zero locus D of s represents the Poincaré dual of the first Chern class $c_1(M)$ and the Maslov index $\mu_L : H_2(M, L; \mathbb{Z}) \rightarrow \mathbb{Z}$ is given by the twice of the intersection number with D . \square

For our purpose, we restrict ourselves to the case that (M, ω) is a closed symplectic manifold, J is an almost complex structure compatible with ω and $L \subset X$ is an oriented Lagrangian submanifold such that $\mu(\beta) \geq 2$ if the moduli space $\mathcal{M}(L; J; \beta) \neq \emptyset$ of bordered stable maps in the class $\beta \neq 0$. See [FOOO5] Appendix 1 for related results under this condition. The following theorem was proved (in Fano toric case) by Auroux [Au] Theorem 6.1.

Theorem 23.13. *Let \mathfrak{b} be a cycle of codimension 2 in M with coefficients in Λ_+ and $b \in \mathcal{M}_{\text{weak, def}}(L; \mathfrak{b})$. Then, for any cycle A in M , we have*

$$i_{\text{qm}, (\mathfrak{b}, b)}^*(c_1(M) \cup_{\mathfrak{b}} PD(A)) = \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}(b) i_{\text{qm}, (\mathfrak{b}, b)}^*(PD(A)) \quad (23.8)$$

in $HF((L, \mathfrak{b}, b); \Lambda)$.

Proof. Let D be the cycle in $M \setminus L$ obtained in Lemma 23.12. Since $c_1(M)$ is the Poincaré dual of D as a cycle in M , we use D to prove the formula (23.8).

The strategy of the proof is the same as in the one of Theorem 9.1 in [FOOO6]. Let $\mathcal{M}_{\ell+2;k+1}$ be the moduli space of genus zero bordered stable curves with $k+1$ boundary marked points z_0, \dots, z_k and $\ell+2$ interior marked points $z_1^+, \dots, z_{\ell+2}^+$ with the connected boundary. Let $\mathcal{M}_{\ell+2;k+1}(\beta)$ be the moduli space of bordered stable maps from genus zero bordered semi-stable curves with $k+1$ boundary marked points and $\ell+2$ interior marked points with the connected boundary to (M, L) representing the class $\beta \in H_2(M, L; \mathbb{Z})$. Here the boundary marked points are ordered in the counter-clockwise way. We denote by $\text{ev}^+ = (\text{ev}_1^+, \dots, \text{ev}_{\ell+2}^+)$ the evaluation map at the interior marked points and by ev_i the evaluation map at the boundary marked point z_i . We set $\text{ev} = (\text{ev}_1, \dots, \text{ev}_k)$.

For cycles $Q_1, \dots, Q_{\ell+2}$ in M and chains P_1, \dots, P_k in L , we define

$$\begin{aligned} & \mathcal{M}_{k+1, \ell+2}(\beta; Q_1 \otimes \dots \otimes Q_{\ell+2}; P_1, \dots, P_k) \\ &:= \mathcal{M}_{k+1, \ell+2}(\beta)_{\text{ev}^+ \times \text{ev}} \times_{M^{\ell+2} \times L^k} (Q_1 \times \dots \times Q_{\ell+2} \times P_1 \times \dots \times P_k). \end{aligned}$$

By taking the stablized domain of the stable map and forgetting the boundary marked points z_1, \dots, z_k and the interior marked points $z_3^+, \dots, z_{\ell+2}^+$, we obtain the forgetful map

$$\text{forget} : \mathcal{M}_{k+1, \ell+2}(\beta; D \otimes A \otimes \mathfrak{b}^{\otimes \ell}; b^{\otimes k}) \rightarrow \mathcal{M}_{1;2}.$$

The moduli space $\mathcal{M}_{1;2}$ of bordered stable curves of genus 0, connected boundary with two interior marked points and one boundary marked point is of complex dimension 1. We pick two points $[\Sigma_0], [\Sigma_1]$ in $\mathcal{M}_{1;2}$ as follows. The bordered stable curve Σ_0 is the union of the unit disc with $z_0 = 1$ on its boundary and the Riemann sphere with z_1^+, z_2^+ , which are away from the interior node of Σ_0 . The bordered stable curve Σ_1 consists of the union of two copies D_0, D_1 of the unit disc with a boundary node such that $z_0 = 1, a_1^+ = 0$ in $D_0, z_2^+ = 0$ in D_1 and the boundary node corresponds to $-1 \in \partial D_0, 1 \in \partial D_1$.

In our case, since the Maslov index $\mu(\beta)$ is at least 2 if $\mathcal{M}(L; J; \beta) \neq \emptyset$, and \mathfrak{b} is represented by codimension 2 cycle, it is enough to study holomorphic discs of Maslov index 2 for the computation of $\mathfrak{m}_0^{\mathfrak{b}, b}(1)$. Recall also that the Maslov class of $L \subset M$ is equal to the twice of the intersection number with D . Therefore

$$\begin{aligned} \mathfrak{q}([D \otimes e^{\mathfrak{b}}]; e^b) &:= \mathfrak{q} \left(\sum_{\ell_1, \ell_2} \frac{1}{(\ell_1 + \ell_2 + 1)!} \mathfrak{b}^{\otimes \ell_1} \otimes D \otimes \mathfrak{b}^{\otimes \ell_2}; e^b \right) \\ &= \mathfrak{q}(e^{\mathfrak{b}}; e^b) = \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}(b) \cdot 1 \end{aligned}$$

for $b \in \mathcal{M}_{\text{def, weak}}(L, \mathfrak{b})$. Hence we find that the sum of contributions from $\text{ev}_0 : \text{forget}^{-1}([\Sigma_1]) \rightarrow L$ is equal to $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}}(b) i_{\text{qm}, (\mathfrak{b}, b)}^*(PD(A))$. On the other hand, we find that $\text{ev}_0 : \text{forget}^{-1}([\Sigma_0]) \rightarrow L$ contributes to $i_{\text{qm}, (\mathfrak{b}, b)}^*(c_1(M) \cup_{\mathfrak{b}} PD(A))$. Now Theorem 23.13 follows in a way similar to [FOOO6] Theorem 9.1. \square

Corollary 23.14. *If \mathbf{A} is an eigenvector of $c_1(M) \cup^{\mathfrak{b}}$ on $QH_{\mathfrak{b}}(M; \Lambda)$ with eigenvalue λ and $i_{\text{qm}, (\mathfrak{b}, b)}^*(\mathbf{A}) \neq 0$, then $\lambda = \mathfrak{P}\mathfrak{D}_{\mathfrak{b}}(b)$.*

We return to the discussion on $T(\rho)$. For $T(0) = T(\rho = 0)$, we can find that the potential function (without bulk deformations) of $T(0)$ has two critical points with critical values $\pm 4T^{1/2}$, by the result of the calculation in [FOOO5] Digression

4.1, where $T(0)$ is denoted by $T(\mathbf{u}_0)$. We have two bounding cochains b_1, b_2 with critical values $4T^{1/2}, -4T^{1/2}$ up to gauge equivalence.

Theorem 3.8.62 in [FOOO1] with taking (3.8.36.2) in Theorem 3.8.32 and Lemma 23.5 and Lemma 23.3 (1) into account implies that

$$i_{\text{qm}, S_{\text{van}}^2}^* : H(S^2 \times S^2; \Lambda) \rightarrow HF(S_{\text{van}}^2; \Lambda),$$

resp.

$$i_{\text{qm}, T(0), b_i}^* : H(S^2 \times S^2; \Lambda) \rightarrow HF((T(0), b_i); \Lambda)$$

sends $\sum_{j=1}^4 \mathbf{e}_j$ to the unit $PD[S_{\text{van}}^2] \neq 0$ of $HF(S_{\text{van}}^2; \Lambda)$, resp. the unit $PD[T(0)] \neq 0$ of $HF((T(0), b_i); \Lambda)$, $i = 1, 2$.

Recall that $QH(S^2 \times S^2; \Lambda)$ is semi-simple and decomposes into $\bigoplus_{i=1}^4 \Lambda \mathbf{e}_i$. We may assume that $\mathbf{e}_1, \mathbf{e}_4$ are eigenvectors of the quantum multiplication by $c_1(S^2 \times S^2)$ with eigenvalues $\pm 4T^{1/2}$ and $\mathbf{e}_2, \mathbf{e}_3$ are those with eigenvalue 0. Comparing the critical values of the potential function and eigenvalues of the quantum multiplication by $c_1(S^2 \times S^2)$, Theorem 23.13 implies that

$$\begin{aligned} i_{\text{qm}, S_{\text{van}}^2}^*(\mathbf{e}_2 + \mathbf{e}_3) &= PD[S_{\text{van}}^2], \\ i_{\text{qm}, T(0), b_1}^*(\mathbf{e}_1) &= PD[T(0)], \\ i_{\text{qm}, T(0), b_2}^*(\mathbf{e}_4) &= PD[T(0)]. \end{aligned}$$

We may assume that $i_{\text{qm}, S_{\text{van}}^2}^*(\mathbf{e}_2) \neq 0$. By Theorem 18.8 (2), we find that S_{van}^2 is $\mu_{\mathbf{e}_2}$ -superheavy and while $T(0)$ is $\mu_{\mathbf{e}_1}$ -superheavy and $\mu_{\mathbf{e}_4}$ -superheavy. On the other hand, since S_{van}^2 and $T(0)$ are disjoint, two quasimorphisms corresponding to $\mu_{\mathbf{e}_1}$ and $\mu_{\mathbf{e}_2}$ are distinct by Theorem 18.7 (and Remark 18.6). This statement is mentioned without proof in [FOOO5] Remark 7.1.

As we showed in Subsection 23.2, the potential function of $T(0)$ with bulk deformation by \mathbf{b}_ρ has two critical points $(\epsilon(e^{T^{\rho/2}} + e^{-T^{\rho/2}})^{-1}, \epsilon(e^{T^{\rho/2}} + e^{-T^{\rho/2}}))$ with $\epsilon = \pm 1$. The critical values are $\pm 2(e^{T^{\rho/2}} + e^{-T^{\rho/2}})T^{1/2}$.

For $T(\rho)$, the potential function with bulk \mathbf{b}_ρ has critical points $(\epsilon T^\rho(e^{T^{\rho/2}} - e^{-T^{\rho/2}})^{-1}, -\epsilon T^{-\rho}(e^{T^{\rho/2}} - e^{-T^{\rho/2}}))$. The critical values are $\pm 2(e^{T^{\rho/2}} - e^{-T^{\rho/2}})T^{1/2}$.

Theorem 3.8.62 in [FOOO1] with taking (3.8.36.2) in Theorem 3.8.32 and Lemma 23.3 (2), (3) into account implies that there exist $b_i \in H^1(T(0); \Lambda_0)$, resp. $b(\rho)_i \in H^1(T(\rho); \Lambda_0)$, $i = 1, 2$ such that

$$i_{\text{qm}, T(0), (\mathbf{b}(\rho), b_i)}^* : QH_{\mathbf{b}(\rho)}(S^2 \times S^2; \Lambda) \rightarrow HF((T(0)(\mathbf{b}(\rho), b_i); \Lambda),$$

$$i_{\text{qm}, T(\rho), (\mathbf{b}(\rho), b(\rho)_i)}^* : QH_{\mathbf{b}(\rho)}(S^2 \times S^2; \Lambda) \rightarrow HF((T(\rho)(\mathbf{b}(\rho), b(\rho)_i); \Lambda)$$

send the unit to the unit.

The eigenvalues of the quantum multiplication with bulk $\mathbf{b}(\rho)$ by $c_1(S^2 \times S^2)$ are as follows. By Remark 5.3 and Theorem 1.9 in [FOOO2] (Fano toric case), Theorem 1.4 in [FOOO6] (general toric case), we find that $\mathbf{e}_1^{\mathbf{b}(\rho)}, \dots, \mathbf{e}_4^{\mathbf{b}(\rho)}$ are eigenvectors of the quantum multiplication by $c_1(S^2 \times S^2)$ with eigenvalues $2(e^{T^{\rho/2}} + e^{-T^{\rho/2}})T^{1/2}$, $2(e^{T^{\rho/2}} - e^{-T^{\rho/2}})T^{1/2}$, $2(-e^{T^{\rho/2}} + e^{-T^{\rho/2}})T^{1/2}$, $-2(e^{T^{\rho/2}} + e^{-T^{\rho/2}})T^{1/2}$, respectively. Hence $b_i, b(\rho)_i$ can be arranged so that

$$i_{\text{qm}, T(0), (\mathbf{b}(\rho), b_1)}^*(\mathbf{e}_1^{\mathbf{b}(\rho)}) = i_{\text{qm}, T(0), (\mathbf{b}(\rho), b_2)}^*(\mathbf{e}_4^{\mathbf{b}(\rho)}) = PD[T(0)]$$

and

$$i_{\text{qm}, T(\rho), (\mathbf{b}(\rho), b(\rho)_1)}^*(\mathbf{e}_2^{\mathbf{b}(\rho)}) = i_{\text{qm}, T(\rho), (\mathbf{b}(\rho), b(\rho)_2)}^*(\mathbf{e}_3^{\mathbf{b}(\rho)}) = PD[T(\rho)].$$

Remark 23.15. The Lagrangian sphere S_{van}^2 is unobstructed without bulk deformation as we saw in Lemma 23.5. Since the self-intersection number of S_{van}^2 is -2 , $\mathfrak{m}_0^{\flat\rho}(1) = -2T^\rho PD[pt]$, it gets obstructed after the bulk deformation by \flat_ρ .

24. LAGRANGIAN TORI IN THE CUBIC SURFACE

This section owes much to the paper [NNU2] of Nishinou-Nohara-Ueda, especially its Subsection 4.1 of the version 1 (arXiv:0812.0066v1). That section contained an error which seems to be a reason why the subsection was removed from the second version (arXiv:0812.0066v2). However, using a result by Chan-Lau [CL], (actually, in [NNU2] Section 5 of the second version they independently obtained the relevant result for the cubic surface by a different argument), we can correct this error. This provides an interesting example which we discuss in this section. We would like to emphasize that the idea of using toric degeneration to calculate the potential function of a non-toric manifold, which we use in this section and in [FOOO5], is due to Nishinou-Nohara-Ueda [NNU1] who applied the idea to various examples successfully.

Following [NNU2] Subsection 4.1 of the version 1, we consider a family of cubic surfaces

$$M_t = \{[x : y : z : w] \in \mathbb{CP}^3 \mid xyz - w^3 = t(x^3 + y^3 + z^3 + w^3)\} \quad (24.1)$$

parametrized by $t \in \mathbb{C}$. For $t \neq 0$ this gives a smooth surface. For $t = 0$, M_0 becomes a toric variety with the $(\mathbb{C}^*)^2$ -action

$$(\alpha, \beta)[x : y : z : w] = [\alpha x : \beta y : \alpha^{-1}\beta^{-1}z : w].$$

The Fubini-Study form on \mathbb{CP}^3 induces a symplectic structure on M_t . This symplectic structure on M_0 is invariant under the action of real torus $T^2 \subset (\mathbb{C}^*)^2$. The moment polytope of this action is given by

$$P = \{(u_1, u_2) \in \mathbb{R}^2 \mid \ell_i(u_1, u_2) \geq 0, i = 1, 2, 3\} \quad (24.2)$$

where

$$\begin{aligned} \ell_1(u_1, u_2) &= -u_1 + 2u_2 + 1, \\ \ell_2(u_1, u_2) &= 2u_1 - u_2 + 1, \\ \ell_3(u_1, u_2) &= -u_1 - u_2 + 1. \end{aligned} \quad (24.3)$$

The moment polytope P is an isosceles triangle, whose center of gravity is origin. The three vertices of P correspond to the three singular points of M_0 . The variety M_0 is a toric orbifold with three singular points of A_2 -type.

We can deform those three singular points by gluing the Milnor fiber of the A_2 singularity by the same way as in Section 23 to obtain a symplectic manifold M . It is easy to see that M is symplectomorphic to M_t for $t \neq 0$. (Note M_t is symplectomorphic to $M_{t'}$ if $t, t' \neq 0$.)

We consider

$$\mathfrak{Z} = (\mathbb{R}_{\geq 0}(1, 0)) \sqcup (\mathbb{R}_{\geq 0}(0, 1)) \sqcup (\mathbb{R}_{\geq 0}(-1, -1)) \cap \text{Int}P.$$

For $\mathbf{u} \in \mathfrak{Z}$ we consider $\pi^{-1}(\mathbf{u}) \subset M_0$. In the same way as in Section 23 we may regard it as a Lagrangian torus in M . We denote it by $T(\mathbf{u})$.

Theorem 24.1. *For each $\mathbf{u} \in \mathfrak{Z}$, there exist $\mathbf{b}(\mathbf{u}) \in H^2(M; \Lambda_+)$ and $b(\mathbf{u}) \in H^1(T(\mathbf{u}); \Lambda_0)$ such that*

$$HF((T(\mathbf{u}), (\mathbf{b}(\mathbf{u}), b(\mathbf{u}))); \Lambda) \neq 0.$$

Moreover there exists $e_{\mathbf{u}}$ that is a unit of a direct product factor $e_{\mathbf{u}}\Lambda = \Lambda$ of $QH_{\mathbf{b}(\mathbf{u})}(M; \Lambda)$ such that

$$i_{\text{qm}, (\mathbf{b}(\mathbf{u}), b(\mathbf{u}))}^*(e_{\mathbf{u}}) \neq 0 \in HF((T(\mathbf{u}), (\mathbf{b}(\mathbf{u}), b(\mathbf{u}))); \Lambda).$$

We can use this theorem in the same way as in Section 23 to show the following.

Corollary 24.2. (1) *Each of $T(\mathbf{u})$ is non-displaceable.*
 (2) *$T(\mathbf{u})$ is not Hamiltonian isotopic to $T(\mathbf{u}')$ if $\mathbf{u} \neq \mathbf{u}'$.*
 (3) *There exist uncountably many homogeneous Calabi quasimorphisms*

$$\mu_{e_{\mathbf{u}}}^{\mathbf{b}(\mathbf{u})} : \widetilde{\text{Ham}}(M; \omega) \rightarrow \mathbb{R}$$

which are linearly independent.

Proof of Theorem 24.1. We consider a toric resolution of our orbifold M_0 , which we denote by $M(\epsilon)$. We may take it so that its moment polytope is

$$P_{\epsilon} = \{(u_1, u_2) \in P \mid \ell_i^{\epsilon}(u_1, u_2) \geq 0, \quad i = 4, \dots, 9\}, \quad (24.4)$$

where

$$\begin{aligned} \ell_4^{\epsilon}(u_1, u_2) &= u_1 + 1 - \epsilon = \frac{1}{3}(2\ell_1 + \ell_2) - \epsilon, \\ \ell_5^{\epsilon}(u_1, u_2) &= u_2 + 1 - \epsilon = \frac{1}{3}(\ell_1 + 2\ell_2) - \epsilon, \\ \ell_6^{\epsilon}(u_1, u_2) &= u_1 - u_2 + 1 - \epsilon = \frac{1}{3}(2\ell_2 + \ell_3) - \epsilon, \\ \ell_7^{\epsilon}(u_1, u_2) &= -u_2 + 1 - \epsilon = \frac{1}{3}(\ell_2 + 2\ell_3) - \epsilon, \\ \ell_8^{\epsilon}(u_1, u_2) &= -u_1 + 1 - \epsilon = \frac{1}{3}(2\ell_3 + \ell_1) - \epsilon, \\ \ell_9^{\epsilon}(u_1, u_2) &= -u_1 + u_2 + 1 - \epsilon = \frac{1}{3}(\ell_3 + 2\ell_1) - \epsilon. \end{aligned} \quad (24.5)$$

We put

$$D_i = \pi^{-1}(\partial_i P_{\epsilon}), \quad \partial_i P_{\epsilon} = \{(u_1, u_2) \in P_{\epsilon} \mid \ell_i^{\epsilon}(u_1, u_2) = 0\},$$

for $i = 4, \dots, 9$. (D_i , $i = 1, 2, 3$ are defined in the same way.)

We note that $M(\epsilon)$ is nef but is not Fano. In fact, $c_1(M(\epsilon)) \cap D_i = 0$ for $i = 4, \dots, 9$. The potential function of $M(\epsilon)$ is calculated by Chan and Lau. In fact, $M(\epsilon)$ is X_{11} in the table given in p.19 of [CL].

Using the fact that M is monotone, we can apply the argument of [FOOO5] Section 6 to show that we can take the limit $\epsilon \rightarrow 0$ to calculate the potential function of $T(\mathbf{u})$ in M . The result is the following.

Let e_1, e_2 be a basis of $H^1(T(\mathbf{u}); \mathbb{Z})$ and put $b = x_1 e_1 + x_2 e_2 \in H^1(T(\mathbf{u}); \Lambda_0)$. We put $\bar{y}_i = e^{x_i}$ and $y_i = T^{u_i} \bar{y}_i$, where $\mathbf{u} = (u_1, u_2)$.

Theorem 24.3. *The potential function of $T(\mathbf{u}) \subset M$ is given by*

$$\begin{aligned} \mathfrak{P}\mathfrak{D} &= T(y_1^{-1} y_2^{-1} (y_1 + y_2)^3 + y_1^{-1} y_2^2 (y_1 y_2^{-1} + y_2^{-1})^3 \\ &\quad + y_1^2 y_2^{-1} (y_1^{-1} + y_1^{-1} y_2)^3 - y_1^{-1} y_2^2 - y_1^2 y_2^{-1} - y_1^{-1} y_2^{-1}). \end{aligned} \quad (24.6)$$

We postpone the proof of Theorem 24.3. We prove it later at the same time as we prove Theorem 24.6, by using (24.14). \square

Corollary 24.4. *For each $\mathbf{u} \in \mathfrak{J}$, there exists $b \in H^1(T(\mathbf{u}), \Lambda_0)$ such that*

$$HF((T(\mathbf{u}), b), (T(\mathbf{u}), b); \Lambda_0) \cong H(T^2; \Lambda_0).$$

Proof. We define Y_1, Y_2 by the formula

$$Y_1^3 = y_1^2 y_2^{-1}, \quad (24.7)$$

$$Y_1^2 Y_2 = y_1. \quad (24.8)$$

Note for each y_1, y_2 there are 3 choices of Y_1 satisfying (24.7). Then (24.8) uniquely determines Y_2 . Thus $(Y_1, Y_2) \mapsto (y_1, y_2)$ is a three to one correspondence.

Now we can rewrite (24.6) as follows:

$$\begin{aligned} \mathfrak{P}\mathfrak{D} &= T \{ (Y_1 + Y_2)^3 + (Y_1 + Y_1^{-1} Y_2^{-1})^3 + (Y_2 + Y_1^{-1} Y_2^{-1})^3 - Y_1^3 - Y_2^3 - Y_1^{-3} Y_2^{-3} \} \\ &= T((Y_1 + Y_2 + Y_1^{-1} Y_2^{-1})^3 - 6). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{T} \frac{\partial \mathfrak{P}\mathfrak{D}}{\partial Y_1} &= 3(1 - Y_1^{-2} Y_2^{-1})(Y_1 + Y_2 + Y_1^{-1} Y_2^{-1})^2 \\ \frac{1}{T} \frac{\partial \mathfrak{P}\mathfrak{D}}{\partial Y_2} &= 3(1 - Y_1^{-1} Y_2^{-2})(Y_1 + Y_2 + Y_1^{-1} Y_2^{-1})^2. \end{aligned}$$

Therefore the critical point is either

$$Y_1 = Y_2, \quad Y_1^3 = 1 \quad (24.9)$$

or

$$Y_1 + Y_2 + Y_1^{-1} Y_2^{-1} = 0. \quad (24.10)$$

(24.9) gives a single solution $y_1 = y_2 = 1$.

(24.10) gives infinitely many critical points. Let us study it a bit more below.

Let $v = v_1 = v_2 < 0$. We put

$$Y_1 = cT^{v_1}, \quad Y_2 = c'T^{v_2}$$

with $\mathfrak{v}_T(c) = \mathfrak{v}_T(c') = 0$. For simplicity we also assume $c \in \mathbb{C} \setminus \{0\}$. (24.10) implies that $c' + c \equiv 0 \pmod{\Lambda_+}$. We put

$$c' = -c(1 + \alpha), \quad \alpha \in \Lambda_+.$$

Then (24.10) becomes

$$-c^3 \alpha(1 + \alpha) = T^{-3v}.$$

Therefore we get

$$\alpha = T^{-3v}(-c^{-3} + \dots),$$

where \dots is an element of Λ_+ . Thus

$$y_1 = Y_1^2 Y_2 = -c^3 T^{3v}(1 - c^{-3} T^{-3v} + \dots),$$

$$y_2 = Y_1 Y_2^2 = c^3 T^{3v}(1 - 2c^{-3} T^{-3v} + \dots),$$

is a critical point. Namely there exists a critical point whose valuation is $\mathbf{u} = (3v, 3v) \in \mathbb{R}_{>0}(-1, -1)$, for any $v < 0$. The corollary now follows from an obvious \mathbb{Z}_3 symmetry. \square

Remark 24.5. Corollary 24.4 implies that the Jacobian ring

$$\text{Jac}(\mathfrak{P}\mathfrak{D}; \Lambda) = \frac{\Lambda \langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}}{\text{Clos}_{d_{\mathring{P}}} \left(y_i \frac{\partial \mathfrak{P}\mathfrak{D}}{\partial y_i}; i = 1, 2 \right)}$$

is infinite dimensional over Λ . In the toric case it is always finite dimensional since

$$\mathfrak{k}_{\mathfrak{s}_0} : QH(X; \Lambda) \rightarrow \text{Jac}(\mathfrak{P}\mathfrak{D}; \Lambda)$$

is an isomorphism. (Theorem 20.18.)

Corollary 24.4 implies the existence of a continuum of mutually disjoint non-displaceable Lagrangian tori in a cubic surface. To show the existence of infinitely many Calabi quasimorphisms and prove Theorem 1.11 (2), we need to study bulk deformations. Let

$$\vec{w} = (w_1, \dots, w_9) \in \Lambda_0^9. \quad (24.11)$$

We put

$$\mathfrak{b}(\vec{w}) = \sum_{i=1}^9 w_i PD(D_i). \quad (24.12)$$

Theorem 24.6. *We have*

$$\begin{aligned} & \frac{1}{T}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})} - \mathfrak{P}\mathfrak{D}) \\ &= (e^{w_1} - 1)y_1^{-1}y_2^2 + (e^{w_2} - 1)y_1^2y_2^{-1} + (e^{w_3} - 1)y_1^{-1}y_2^{-1} \\ & \quad + (e^{w_4} + e^{w_5-w_4} + e^{-w_5} - 3)y_1 + (e^{w_5} + e^{w_4-w_5} + e^{-w_4} - 3)y_2 \\ & \quad + (e^{w_6} + e^{w_7-w_6} + e^{-w_7} - 3)y_1y_2^{-1} + (e^{w_7} + e^{w_6-w_7} + e^{-w_6} - 3)y_2^{-1} \\ & \quad + (e^{w_8} + e^{w_9-w_8} + e^{-w_9} - 3)y_1^{-1} + (e^{w_9} + e^{w_8-w_9} + e^{-w_8} - 3)y_1^{-1}y_2. \end{aligned}$$

Proof. We consider the term $3y_1^{-1}y_2^{-1}y_1^2y_2 = 3y_1$ in (24.6). This term comes from the moduli space $\mathcal{M}(\beta)$ where $\beta = \beta_4 + \alpha$ with

$$\beta_4 \cap D_j = \begin{cases} 1 & j = 4 \\ 0 & j \neq 4, \end{cases} \quad (24.13)$$

and $\alpha \in H_2(M; \mathbb{Z})$ with

$$\alpha = k_1[D_4] + k_2[D_5].$$

We define

$$d(k_1, k_2) = \deg(\text{ev}_0 : \mathcal{M}_1(\beta_4 + k_1[D_4] + k_2[D_5]) \rightarrow L(\mathbf{u})).$$

By the result of Chan-Lau, [CL] Theorem 1.1, (and the fact that the potential functions are continuous with respect to the limit $\epsilon \rightarrow 0$), we derive

$$d(k_1, k_2) = \begin{cases} 1 & (k_1, k_2) = (0, 0), (1, 0), (1, 1) \\ 0 & \text{otherwise.} \end{cases} \quad (24.14)$$

This result is also obtained independently in Section 5 of the second version of [NNU2] based on the another argument. Therefore, by the proof of [FOOO3] Proposition 9.4, the coefficient of y_1 in $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}_{a,b}(u)}$ is given by

$$\begin{aligned} & \sum_{k_1, k_2} d(k_1, k_2) \exp(w_4[D_4] \cap [\beta_4 + k_1[D_4] + k_2[D_5]]) \\ & \quad \exp(w_5[D_5] \cap [\beta_4 + k_1[D_4] + k_2[D_5]]) \\ &= e^{w_4} + e^{w_5-w_4} + e^{-w_5}. \end{aligned}$$

(Here we use (24.13) and $[D_4] \cdot [D_4] = [D_5] \cdot [D_5] = -2$, $[D_4] \cdot [D_5] = 1$.)

In the same way the coefficient of y_2 in $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}$ is given by $e^{w_5} + e^{w_4-w_5} + e^{-w_4}$. This proves the second line of the right hand side. The third and fourth line can be proved in the same way. The proof of the first line is easier. \square

We put

$$\vec{w}_0 = (0, 0, 0, w_0, w_0, w_0, w_0, w_0, w_0, w_0), \quad e^{2w_0} + e^{w_0} + 1 = 0.$$

Theorem 24.6 implies

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w}_0)} = T(y_1^{-1}y_2^2 + y_1^2y_2^{-1} + y_1^{-1}y_2^{-1}). \quad (24.15)$$

Remark 24.7. According to [Iri2] Proposition 3.10, (24.15) is the Laudau-Ginzburg superpotential of the mirror of the toric orbifold M_0 .

Lemma 24.8. $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w}_0)}$ has 9 critical points. All of them have valuation 0 and nondegenerate.

Proof. We can easily check that the critical points are given by $y_1^3 = y_2^3 = 1$. \square

Lemma 24.9. For the generic \vec{w} the set of critical points of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}$ consists of 9 elements all of which are nondegenerate.

Proof. The Newton polytope of the Laurent polynomial $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}$ of y has volume $9/2$. Therefore by the result of Kushnirenko [Ku] the number of critical points are at most 9. Since it is exactly 9 in case of $\vec{w} = \vec{w}_0$, it is so for generic \vec{w} . Since the number is maximal it must be nondegenerate. \square

Lemma 24.10. Suppose that the set of critical points of $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}$ consists of 9 elements all of which are nondegenerate. We assume also that the valuation of the critical points are in the interior of the moment polytope P . We also assume that none of the following happens.

- (1) $e^{3w_4} = 1, e^{2w_4} = e^{w_5}$.
- (2) $e^{3w_6} = 1, e^{2w_6} = e^{w_7}$.
- (3) $e^{3w_8} = 1, e^{2w_8} = e^{w_9}$.

Then the homomorphism

$$\mathfrak{fs}_{\mathfrak{b}(\vec{w})} : QH_{\mathfrak{b}(\vec{w})}(X; \Lambda) \rightarrow \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}; \Lambda)$$

is an isomorphism. Moreover $QH_{\mathfrak{b}(\vec{w})}(X; \Lambda)$ is semi-simple.

Proof. We can prove that $\mathfrak{fs}_{\mathfrak{b}(\vec{w})}$ is a ring homomorphism in a similar way as in [FOOO6] Theorem 9.1. (See [AFOOO] for detail.) We put

$$z_i = y_1^{\frac{\partial \ell_i}{\partial y_1}} y_2^{\frac{\partial \ell_i}{\partial y_2}}.$$

In a way similar to [FOOO6] Lemma 2.3, we can prove that $\{z_i \mid i = 1, \dots, 9\}$ generates a dense Λ -subalgebra of $\Lambda\langle\langle y, y^{-1} \rangle\rangle^{\mathring{P}}$. Since $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}; \Lambda)$ is finite dimensional, it is generated by the image of $z_i, i = 1, \dots, 9$ as Λ -algebra.

By differentiating the formula given in Theorem 24.6, we find that, for each $i = 1, 2, 3$, the cohomology class $PD[D_i]$ is mapped to $e^{w_i} z_i$ by $\mathfrak{fs}_{\mathfrak{b}(\vec{w})}$. We calculate

$$\begin{pmatrix} \mathfrak{fs}_{\mathfrak{b}(\vec{w})}(PD([D_4])) \\ \mathfrak{fs}_{\mathfrak{b}(\vec{w})}(PD([D_5])) \end{pmatrix} = \begin{pmatrix} e^{w_4} - e^{w_5-w_4} & e^{w_4-w_5} - e^{-w_4} \\ e^{-w_4+w_5} - e^{-w_5} & e^{w_5} - e^{w_4-w_5} \end{pmatrix} \begin{pmatrix} z_4 \\ z_5 \end{pmatrix}$$

By assumption the matrix in the right hand side is nonzero. Therefore the image of $\mathfrak{fs}_{\mathfrak{b}(\vec{w})}$ contains either z_4 or z_5 . Since $z_4 z_5 = z_1 z_2$, it contains both of z_4 and z_5 . In a similar way we find that the image of $\mathfrak{fs}_{\mathfrak{b}(\vec{w})}$ contains z_6, \dots, z_9 . Therefore $\mathfrak{fs}_{\mathfrak{b}(\vec{w})}$ is surjective.

The rank of $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w})}; \Lambda)$ is 9 that is equal to the Betti number of X . Therefore $\mathfrak{fs}_{\mathfrak{b}(\vec{w})}$ is an isomorphism. Therefore $QH_{\mathfrak{b}(\vec{w})}(X; \Lambda) \cong \Lambda^9$ is semi-simple. \square

We next put

$$\vec{w}_{u;c} = (0, 0, 0, w(u; c), w(u; c), 0, 0, 0, 0), \quad e^{w(u;c)} + 1 + e^{-w(u;c)} = 3 + cT^u$$

with $c \in \mathbb{C} \setminus \{0\}$, $u \geq 0$. (We observe that $\mathfrak{v}_T(w(u; c)) = u/2$.)

Lemma 24.11. *We assume c is generic in case $u = 0$.*

- (1) $\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w}_{u;c})}$ has exactly 3 nonzero critical points.
- (2) The valuation of the critical points are $(0, 0)$.
- (3) All the three critical points are nondenerate.

Proof. We take variables Y_1, Y_2 as in (24.7), (24.8). Then by Theorem 24.6 we have

$$\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w}_{u;c})} = T((Y_1 + Y_2 + Y_1^{-1}Y_2^{-1})^3 - 6) + cT^{1+u}(Y_1 + Y_2)Y_1Y_2.$$

Therefore $\nabla \mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\vec{w}_{u;c})} = 0$ is equivalent to

$$3(1 - Y_1^{-2}Y_2^{-1})(Y_1 + Y_2 + Y_1^{-1}Y_2^{-1})^2 + cT^uY_2(2Y_1 + Y_2) = 0, \quad (24.16)$$

$$3(1 - Y_1^{-1}Y_2^{-2})(Y_1 + Y_2 + Y_1^{-1}Y_2^{-1})^2 + cT^uY_1(2Y_2 + Y_1) = 0. \quad (24.17)$$

Therefore

$$\frac{2Y_1 + Y_2}{Y_1^2Y_2 - 1} = \frac{2Y_2 + Y_1}{Y_2^2Y_1 - 1}.$$

Using $Y_1 \neq 0, Y_2 \neq 0$ it implies either $Y_1 = Y_2$ or $Y_1Y_2(Y_1 + Y_2) = -1$.

In case $Y_1Y_2(Y_1 + Y_2) = -1$ we use (24.16) to find $2Y_1 + Y_2 = 0$. We also use (24.17) to find $2Y_2 + Y_1 = 0$. This is impossible. Thus we have $Y_1 = Y_2$. We put $x = Y_1^3 = Y_2^3$. Then (24.16), (24.17) are equivalent to

$$(x - 1)(2x + 1)^2 + cT^ux^3 = 0. \quad (24.18)$$

This equation has three simple roots. (We use genericity of c in case $u = 0$.) We have proved (1). (Note $y_1 = y_2 = x$.)

If $u = 0$, then $x \in \mathbb{C} \setminus \{0\}$. Therefore the valuations of y_1, y_2 are 0. If $u > 0$, then $x \equiv 1$ or $-1/2$ modulo Λ_+ . Therefore $\mathfrak{v}_T(x) = 0$ also. This proves (2).

We next prove (3). We calculate

$$\left(y_i y_j \frac{\partial^2 \mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}}{\partial y_i \partial y_j} \right)_{i,j=1}^2 = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad (24.19)$$

where

$$A = \frac{T}{x^2}(4x^3 + 6x^2 + 6x + 2), \quad B = \frac{T}{x^2}(-4x^3 - 6x^2 + 1).$$

Here we use (24.18) during the calculation. Therefore we get

$$\det \left(y_i y_j \frac{\partial^2 \mathfrak{P}\mathfrak{D}_{\mathfrak{b}(\rho)}}{\partial y_i \partial y_j} \right)_{i,j=1}^2 = 3T^2 \frac{(2x + 1)^4}{x^4}.$$

(3) follows. □

We now consider an affine line $C \cong \mathbb{C}$ contained in \mathbb{C}^9 so that it contains \vec{w}_0 and $\vec{w}_{0;c}$ where c is generic. For $a \in C$ we have $\mathfrak{b}(a)$. We consider

$$\mathfrak{X}_0 = \{(a; y_1, y_2) \mid a \in C, \nabla(\mathfrak{P}\mathfrak{D}_{\mathfrak{b}(a)}) = 0 \text{ at } (y_1, y_2) \in \mathbb{C}^2\}.$$

We take the Zariski closure of \mathfrak{X}_0 in $C \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and denotes it by \mathfrak{X} . We have a projection $\pi : \mathfrak{X} \rightarrow C$. At a generic point $a \in C$ the fiber of π consists of 9 points.

At $a = \vec{w}_{0;c}$ the fiber of π intersects with \mathfrak{X}_0 at 3 points and those three points are all simple. Therefore $\pi^{-1}(\vec{w}_{0;c}) \setminus \mathfrak{X}_0 \neq \emptyset$. Then there exist Laurent power series

$$y_1(w) = \sum_{k \geq k_{0,1}} y_{1,k} w^k, \quad y_2(w) = \sum_{k \geq k_{0,2}} y_{2,k} w^k,$$

and

$$a(w) = \sum_{k \geq 0} a_k w^k$$

such that the following holds:

- (1) $y_1(w), y_2(w)$ converge for $|w| \in (0, \epsilon)$.
- (2) $a(w)$ converges for $|w| \in [0, \epsilon)$.
- (3) $(a(w); y_1(w), y_2(w)) \in \mathfrak{X}_0$ for $|w| \in (0, \epsilon)$.
- (4) $(y_1(0), y_2(0)) \in (\mathbb{C}P^1)^2 \setminus (\mathbb{C} \setminus \{0\})^2$.
- (5) $a(0) = \vec{w}_{0;c}$.

Now we consider $(y_1(T^\rho), y_2(T^\rho)) \in \Lambda^2$ and $a(T^\rho) \in \Lambda_0^9$. (3) implies that $(y_1(T^\rho), y_2(T^\rho))$ is a critical point of $\mathfrak{PD}_{\mathfrak{b}(a(T^\rho))}$.

Lemma 24.12. *If $(\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho))) \in P$, then*

$$(\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho))) \in \mathfrak{Z}.$$

Proof. If $(u_1, u_2) = (\mathfrak{v}_T(y_1), \mathfrak{v}_T(y_2)) \in P \setminus \mathfrak{Z}$, then there exist $i = 1, 2, 3$ such that

$$\ell_i(u_1, u_2) < \ell_j(u_1, u_2)$$

for each $j \in \{1, \dots, 9\}, j \neq i$. It follows easily that (y_1, y_2) is not a critical point of $\mathfrak{PD}_{\mathfrak{b}(a(T^\rho))}$. \square

It is easy to see that $\rho \mapsto (\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho)))$ is continuous and

$$\lim_{\rho \rightarrow 0} (\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho))) = (0, 0).$$

Moreover we find that $\mathfrak{v}_T(y_1(T^\rho))$ and $\mathfrak{v}_T(y_2(T^\rho))$ are either increasing or decreasing, and $(\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho)))$ diverges as $\rho \rightarrow \infty$. (This is a consequence of (4) above.) Therefore there exists $\rho_1 > 0$ such that

$$\rho \mapsto (\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho)))$$

defines a homeomorphism between $(0, \rho_1)$ and one of $\mathfrak{Z}_1 = \{(-u, -u) \mid u \in (0, 1)\}$, $\mathfrak{Z}_2 = \{(u, 0) \mid u \in (0, 1)\}$, $\mathfrak{Z}_3 = \{(0, u) \mid u \in (0, 1)\}$. Note that there exist 6 choices of such $(y_1(q), y_2(q))$ for given $a(w)$, after replacing C by an appropriate branched cover that branches at $w = \vec{w}_{0;c}$. This is because the order of the set $\pi^{-1}(a(w)) \subset \mathfrak{X}$ is 9 for generic w and the set $\pi^{-1}(a(0)) \subset \mathfrak{X}$ consists of 3 points all of which are simple. Each of such 6 choices determines ρ_1 above. We take its minimum and denote it by ρ_0 . Thus we proved the following:

Lemma 24.13. (1) *For each $\rho \in (0, \rho_0)$, there exist exactly 9 critical points of $\mathfrak{PD}_{\mathfrak{b}(a(T^\rho))}$. They are simple and their valuations are always in the interior of P .*

(2) *We may take a choice of $(y_1(q), y_2(q))$ as above such that*

$$\rho \mapsto (\mathfrak{v}_T(y_1(T^\rho)), \mathfrak{v}_T(y_2(T^\rho)))$$

defines a homeomorphism between $(0, \rho_0)$ and one of $\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3$.

Lemmas 24.10, 24.13 and Theorem 18.8 imply that the following holds for one of $i = 1, 2, 3$. We also note that we can prove Theorem 20.23 (20.39) in our situation, where we replace $i_{\text{qm}, (\mathbf{b}(\eta), b(\eta))}^{T,*}$ by $i_{\text{qm}, (\mathbf{b}(a(T^\rho)), b(a(T^\rho)))}^*$.

Lemma 24.14. *For each $\mathbf{u} \in \mathfrak{Z}_i$, there exist $\mathbf{b}(\mathbf{u})$ and $e(\mathbf{u}) \in QH_{\mathbf{b}(\mathbf{u})}(X; \Lambda)$ such that:*

- (1) $e(\mathbf{u})\Lambda \subset QH_{\mathbf{b}(\mathbf{u})}(X; \Lambda)$ is a direct factor.
- (2) $T(\mathbf{u})$ is $\zeta_{e(\mathbf{u})}^{\mathbf{b}(\mathbf{u})}$ -superheavy.

Once we have Lemma 24.14 for some i , then by symmetry we obtain the same conclusion for any $i = 1, 2, 3$. The proof of Theorem 24.1 is now completed. \square

25. DETECTING SPECTRAL INVARIANT VIA HOCHSCHILD COHOMOLOGY

In this section we prove the following theorem. For a critical point η of the potential function $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$ we recall the subset $\text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathbf{b}}; \eta) \subset \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathbf{b}}; \Lambda)$ from Definition 20.21. Corresponding to this subspace, we put

$$QH_{\mathbf{b}}(M; \Lambda; \eta) := \{s \in QH_{\mathbf{b}}(M; \Lambda) \mid \mathfrak{f}\mathfrak{s}_{\mathbf{b}}(s) \in \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathbf{b}}; \eta)\}.$$

Theorem 25.1. *Let (M, ω) be a compact toric manifold and $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0)$. Suppose that η is a critical point of the potential function $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$. Let $\mathbf{u} = \mathbf{u}(\eta)$ and $b = b(\eta)$ those defined as in Theorem 20.17. Denote by e_{η} the idempotent of $QH_{\mathbf{b}}(M, \omega; \eta)$. Then $L(\mathbf{u})$ is $\mu_{e_{\eta}}^{\mathbf{b}}$ -superheavy.*

This theorem improves Theorem 21.1 in that superheavyness holds without assuming nondegeneracy of η .

Problem 25.2. Let η be a degenerate critical point of $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$. When does $\mu_{e_{\eta}}^{\mathbf{b}}$ become a quasimorphism?

25.1. Hochschild cohomology of filtered A_{∞} algebra: review. We use Hochschild cohomology for the proof of Theorem 25.1. Let $(C, \{\mathfrak{m}_k\}_{k=0}^{\infty})$ be a unital and gapped filtered A_{∞} algebra. (See [FOOO1] Section 3 for the definition of filtered A_{∞} algebra etc.) In this section we assume

$$\mathfrak{m}_0 = 0.$$

In the situation of Theorem 25.1, we have $\mathfrak{m}_0(1) = \mathfrak{P}\mathfrak{D}(b) \cdot 1$ that is not zero in general. We *redefine* $\mathfrak{m}_0(1) = 0$ and do not change other operators. By unitarity all the A_{∞} relations still hold. After this modification we apply the argument in this section. We put

$$\begin{aligned} CH^k(C, C) &= \text{Hom}_{\Lambda}(B_k C[1], C[1]), \\ CH(C, C) &= \widehat{\bigoplus_{k=0}^{\infty} CH^k(C, C)}, \\ \mathcal{N}^k CH(C, C) &= CH(C, C) / \widehat{\bigoplus_{k' > k} CH^{k'}(C, C)}. \end{aligned} \tag{25.1}$$

Here $\widehat{\bigoplus}$ is the completion of algebraic direct sum with respect to q -adic topology.

We assume that $(C, \{\mathfrak{m}_k\}_{k=0}^{\infty})$ has a strict unit \mathbf{e} . We define its submodule

$$CH^{\text{red}, k}(C, C) = \{\varphi \in CH^k(C, C) \mid \varphi(\cdots, \mathbf{e}, \cdots) = 0\}$$

and define $CH^{\text{red}}(C, C)$, $\mathcal{N}^k CH^{\text{red}}(C, C)$ in a similar way. We define Hochschild differential $\delta_H : CH(C, C) \rightarrow CH(C, C)$ by

$$\begin{aligned} \delta_H(\varphi)(x_1, \dots, x_k) &= \sum_{i, \ell} (-1)^{*_1} \varphi(x_1, \dots, \mathbf{m}_\ell(x_i, \dots), \dots) \\ &\quad + \sum_{i, \ell} (-1)^{*_2} \mathbf{m}_\ell(x_1, \dots, \varphi(x_i, \dots), \dots), \end{aligned} \quad (25.2)$$

where $*_1 = \deg \varphi(\deg x_1 + \dots + \deg x_{i-1} + i - 1)$, $*_2 = \deg x_1 + \dots + \deg x_{i-1} + i$. It is easy to check $\delta_H \circ \delta_H = 0$. So $(CH(C, C), \delta_H)$ is a (co)chain complex, which we call *Hochschild cochain complex*.

Using our assumption $\mathbf{m}_0 = 0$, we have

$$\delta_H \left(\widehat{\bigoplus_{k' > k} CH^{k'}(C, C)} \right) \subset \widehat{\bigoplus_{k' > k} CH^{k'}(C, C)}.$$

Therefore $\delta_H : \mathcal{N}^k CH(C, C) \rightarrow \mathcal{N}^k CH(C, C)$ is induced. We call this filtration the *number filtration*. Note we have

$$(CH(C, C), \delta_H) = \text{proj} \lim_{k \rightarrow \infty} (\mathcal{N}^k CH(C, C), \delta_H).$$

We can easily show

$$\delta_H(CH^{\text{red}}(C, C)) \subset CH^{\text{red}}(C, C).$$

So the *reduced Hochschild cochain complex* $(CH^{\text{red}}(C, C), \delta_H)$ is defined. It has a number filtration. The cohomology of the reduced Hochschild cochain complex is written as $HH^{\text{red}}(C, C)$ and is called *reduced Hochschild cohomology*.

We note that

$$CH^k(C, C) \cong \text{Hom}_{\mathbb{C}}(B_k \overline{C}[1], \overline{C}[1]) \otimes \Lambda,$$

where $\overline{C} \otimes_{\mathbb{C}} \Lambda = C$. We can then use the filtration $F^\lambda C = \overline{C} \otimes_{\mathbb{C}} T^\lambda \Lambda_0$ to define a filtration $F^\lambda CH^k(C, C)$ on $CH^k(C, C)$. We call this filtration the *energy filtration*. Using the condition

$$\mathbf{m}_k(F^{\lambda_1} C \otimes \dots \otimes F^{\lambda_k} C) \subset F^{\lambda_1 + \dots + \lambda_k} C$$

([FOOO1] (3.2.12.6)), δ_H preserves the energy filtration.

25.2. From quantum cohomology to Hochschild cohomology. Let L be a relative spin Lagrangian submanifold of a compact symplectic manifold (M, ω) and $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0)$. Let $b = b_0 + b_+ \in \Omega^{\text{odd}} \otimes \Lambda_0$ be an element satisfying the Maurer-Cartan equation

$$\sum_{\beta} \sum_{k=0}^{\infty} T^{\omega \cap \beta} \exp(b_0 \cap \partial \beta) \mathbf{m}_{k, \beta}(b_+, \dots, b_+) = 0.$$

For each such pair (\mathbf{b}, b) , we obtain a unital and gapped filtered A_∞ algebra $(\Omega(L) \widehat{\otimes} \Lambda, \{\mathbf{m}_k^{\mathbf{b}, b}\})$. We define

$$\mathbf{q}_*^{\mathbf{b}} = H(M; \Lambda) \rightarrow CH^{\text{red}}(\Omega(L) \widehat{\otimes} \Lambda, \Omega(L) \widehat{\otimes} \Lambda) \quad (25.3)$$

as follows. We put $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+)$ and define for $k \neq 0$:

$$\begin{aligned} & \mathbf{q}_*^{\mathbf{b}}(a)(x_1, \dots, x_k) \\ &= \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sum_{m_0=0}^{\infty} \dots \sum_{m_k=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{(\ell_1 + \ell_2 + 1)!} \\ & \mathbf{q}_{\ell_1 + \ell_2 + 1, k + \sum_{i=0}^k m_i; \beta}(\mathbf{b}_+^{\otimes \ell_1} a \mathbf{b}_+^{\otimes \ell_2}; b_+^{\otimes m_0}, x_1, b_+^{\otimes m_1}, \dots, b_+^{\otimes m_{k-1}}, x_k, b_+^{\otimes m_k}), \end{aligned} \quad (25.4)$$

where $x_i \in \Omega(L)$. Here the notations are as in (17.11). Recall that we assume $\mathbf{m}_0 = 0$ in this section.

Using Theorem 17.1 (1), we can show that $\delta_H(\mathbf{q}_*^{\mathbf{b}}(a)) = 0$. Therefore we have a map

$$\mathbf{q}_*^{\mathbf{b}} : H(M; \Lambda) \rightarrow HH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda) \quad (25.5)$$

to the Hochschild cohomology. Here $HH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda)$ is the Hochschild cohomology with respect to the filtered A_{∞} structure $\mathbf{m}^{\mathbf{b}}$.

Remark 25.3. (1) By composing $\mathbf{q}_*^{\mathbf{b}}$ with the projection

$$HH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda) \rightarrow \mathcal{N}_0 HH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda) = HF^*((L, \mathbf{b}); \Lambda),$$

we obtain a map

$$H(M; \Lambda) \rightarrow HF^*((L, \mathbf{b}); \Lambda).$$

This coincides with the map $i_{\text{qm}, HF((L, \mathbf{b}); \Lambda)}^*$ given in (17.18).

(2) $HH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda)$ has a filtered A_{∞} structure. (See [FOOO6] (31.4) and Remark 31.1 (1).) Moreover $\mathbf{q}_*^{\mathbf{b}}$ is a ring homomorphism.

25.3. Proof of Theorem 25.1. To prove Theorem 25.1, we need to explore some estimates of the spectral invariant which are analogs of ones developed in Chapters 1-4. In the previous chapters we use Λ^{\downarrow} coefficients and the valuation \mathbf{v}_q , while we use Λ coefficients and the valuation \mathbf{v}_T in Subsections 25.1, 25.2. To translate the valuation \mathbf{v}_T for any element x defined over Λ into \mathbf{v}_q , we just define

$$\mathbf{v}_q(x) = -\mathbf{v}_T(x),$$

because $T = q^{-1}$. See Notations and Conventions (16) in Section 1. We use this notation throughout this subsection.

The following is an analog of Proposition 18.9.

Proposition 25.4. *Let L, \mathbf{b} be as above and $a \in H(M; \Lambda)$. Then*

$$\rho^{\mathbf{b}}(H; a) \geq \inf\{-H(t, p) \mid (t, p) \in S^1 \times L\} + \mathbf{v}_q(\mathbf{q}_*^{\mathbf{b}}(a)).$$

Here for $x \in HH^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda)$ we define

$$\mathbf{v}_q(x) = -\mathbf{v}_T(x) = -\sup\{\lambda \mid \exists \tilde{x} \in F^{\lambda} CH^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda), x = [\tilde{x}]\}.$$

Proof. Using the operator $\mathbf{q}_{\beta; \ell; k}^F$ in (18.17), we define

$$\mathbf{q}_*^{F, \mathbf{b}} : CF(M; H; \Lambda) \rightarrow CH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda) \quad (25.6)$$

by

$$\begin{aligned}
& \mathfrak{q}_*^{F,\mathbf{b}}([\gamma, w])(x_1, \dots, x_k) \\
&= \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell=0}^{\infty} \sum_{m_0=0}^{\infty} \dots \sum_{m_k=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{\ell!} \\
& \quad \mathfrak{q}_{\ell, k + \sum_{i=0}^k m_i; \beta}^F(\mathbf{b}_+^{\otimes \ell}; [\gamma, w]; b_+^{\otimes m_0}, x_1, b_+^{\otimes m_1}, \dots, b_+^{\otimes m_{k-1}}, x_k, b_+^{\otimes m_k}).
\end{aligned} \tag{25.7}$$

Using Proposition 18.15, we can find that it induces a map

$$\mathfrak{q}_*^{F,\mathbf{b}} : HF(M, H; \Lambda) \rightarrow HH_{\mathbf{b}}^{\text{red}}(\Omega(L) \hat{\otimes} \Lambda, \Omega(L) \hat{\otimes} \Lambda).$$

Lemma 25.5. $\mathfrak{q}_*^{F,\mathbf{b}} \circ \mathcal{P}_{(H_{\chi}, J)}^{\mathbf{b}}$ is chain homotopic $\mathfrak{q}_*^{\mathbf{b}}$.

The proof is the same as that of Proposition 18.21 and is omitted.

We can use Lemma 25.5 to prove Proposition 25.4 in the same way as we used Proposition 18.21 to prove Proposition 18.9. Thus Proposition 25.4 follows. \square

Now we consider the case of toric manifold (M, ω) . For the toric case we can use \mathfrak{q}^T in place of \mathfrak{q} in Proposition 25.4 and the Hochschild complex

$$CH(H(L(\mathbf{u}); \Lambda), H(L(\mathbf{u}); \Lambda))$$

defined on de Rham cohomology (instead on the space of differential forms). In fact, \mathfrak{m}_*^T has been defined on it.

Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda)$ and let \mathfrak{y} be a critical point of $\mathfrak{P}\mathfrak{D}_{\mathbf{b}}$, $\mathbf{u} = \mathbf{u}(\mathfrak{y})$ and $b = b(\mathfrak{y})$ as in Theorem 20.17. We put $\mathbf{b} = \mathbf{b}(\mathfrak{y}) = (\mathbf{b}, b(\mathfrak{y}))$.

Lemma 25.6. The restriction of $\mathfrak{q}_*^{T,\mathbf{b}}$ to $QH_{\mathbf{b}}(M; \Lambda; \mathfrak{y}) \subset QH_{\mathbf{b}}(M; \Lambda)$,

$$\mathfrak{q}_*^{T,\mathbf{b}} : QH_{\mathbf{b}}(M; \Lambda; \mathfrak{y}) \rightarrow HH_{\mathbf{b}}^{\text{red}}(H(L(\mathbf{u}); \Lambda), H(L(\mathbf{u}); \Lambda))$$

is injective.

Proof. By [FOOO6] Lemma 31.5 there exists a map

$$HH_{\mathbf{b}}^{\text{red}}(H(L(\mathbf{u}); \Lambda), H(L(\mathbf{u}); \Lambda)) \rightarrow \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathbf{b}}; \mathfrak{y}). \tag{25.8}$$

The composition of the restriction of $\mathfrak{q}_*^{\mathbf{b}}$ to $QH_{\mathbf{b}}(M; \Lambda; \mathfrak{y})$ with (25.8) is an isomorphism $QH_{\mathbf{b}}(M; \Lambda; \mathfrak{y}) \cong \text{Jac}(\mathfrak{P}\mathfrak{D}_{\mathbf{b}}; \mathfrak{y})$. Hence the lemma. \square

Since the image of the map $\mathfrak{q}_*^{T,\mathbf{b}}$ is a finite dimensional vector space over Λ , we can apply the argument of Subsection 8.1 to find a standard basis $\mathfrak{q}_*^{T,\mathbf{b}}(e_1), \dots, \mathfrak{q}_*^{T,\mathbf{b}}(e_k)$ of the image of $\mathfrak{q}_*^{T,\mathbf{b}}$. Then we have

$$\begin{aligned}
\mathfrak{v}_q \left(\mathfrak{q}_*^{T,\mathbf{b}} \left(\sum_{i=1}^k x_i e_i \right) \right) &= \max \{ \mathfrak{v}_q(x_i \mathfrak{q}_*^{T,\mathbf{b}}(e_i)) \mid i = 1, \dots, k \} \\
&\geq \max \{ \mathfrak{v}_q(x_i) \mid i = 1, \dots, k \} - C_1
\end{aligned} \tag{25.9}$$

where C_1 is independent of x_i .

Now we are ready to complete the proof of Theorem 25.1. Let $\tilde{\psi}_H \in \widetilde{\text{Ham}}(M, \omega)$. By Theorem 15.1 we have

$$\rho^{\mathbf{b}}(\tilde{\psi}_H^n; e_{\mathfrak{y}}) = - \inf \{ \rho^{\mathbf{b}}(\tilde{\psi}_H^{-n}; b) \mid \Pi(e_{\mathfrak{y}}, b) \neq 0 \}. \tag{25.10}$$

Let us estimate the right hand side of (25.10). Suppose $\Pi(e_\eta, b) \neq 0$. We put

$$e_\eta \cup^b b = \sum_{i=1}^k x_i e_i, \quad x_i \in \Lambda.$$

Since $\Pi(e_\eta, b) \neq 0$, Sublemma 16.6 implies $\mathbf{v}_q(e_\eta \cup^b b) \geq 0$. Therefore

$$\max\{\mathbf{v}_q(x_i) \mid i = 1, \dots, k\} \geq C_2, \quad (25.11)$$

where $C_2 = -\max\{\mathbf{v}_q(e_i) \mid i = 1, \dots, k\}$.

By triangle inequality,

$$\rho^b(\tilde{\psi}_H^{-n}; b) \geq \rho^b(\tilde{\psi}_H^{-n}; \sum_{i=1}^k x_i e_i) - \rho^b(\mathbb{Q}; e_\eta).$$

Using Proposition 25.4 the right hand side is not smaller than

$$n \inf\{-\tilde{H}(t, p) \mid (t, p) \in S^1 \times M\} + \mathbf{v}_q \left(\mathbf{q}_*^{T, b} \left(\sum_{i=1}^k x_i e_i \right) \right) - \rho^b(\mathbb{Q}; e_\eta).$$

By (25.9), this is not smaller than

$$n \inf\{H(t, p) \mid (t, p) \in S^1 \times M\} + \max\{\mathbf{v}_q(x_i) \mid i = 1, \dots, k\} - C_1 - \rho^b(\mathbb{Q}; e_\eta).$$

Using (25.11) we have

$$\rho^b(\tilde{\psi}_H^{-n}; b) \geq n \inf\{H(t, p) \mid (t, p) \in S^1 \times M\} + C_2 - C_1 - \rho^b(\mathbb{Q}; e).$$

Therefore by (25.10), we obtain

$$-\frac{\rho^b(\tilde{\psi}_H^n; e_\eta)}{n} \geq \inf\{H(t, p) \mid (t, p) \in S^1 \times M\} - \frac{C_3}{n},$$

where C_3 is independent of n . Therefore we obtain

$$\begin{aligned} \frac{\rho^b(\tilde{\psi}_H^n; e_\eta)}{n} &\leq -\inf\{H(t, p) \mid (t, p) \in S^1 \times M\} + \frac{C_3}{n} \\ &= \sup\{-H(t, p) \mid (t, p) \in S^1 \times M\} + \frac{C_3}{n}. \end{aligned}$$

By letting $n \rightarrow \infty$, we have finished the proof of Theorem 25.1. \square

25.4. A remark. In Theorems 21.1 and 25.1, we use Lagrangian Floer theory to estimate the spectral invariant in terms of the values of the Hamiltonian on the Lagrangian submanifolds. One can use a variant of this technique to obtain an estimate of spectral invariant using various other invariant appearing in symplectic topology.

By using the Hamiltonian $H = H(t, x)$ itself as Albers did in [Al] instead of τ -dependent modification F we use in Subsection 18.2, we can improve the statement of Proposition 18.9 to the following

$$\rho^b(H; a) \geq -E^+(H; \mathcal{L}(Y)) + \rho_L^b(a) \quad (25.12)$$

where the invariant $E^+(H; \mathcal{L}(Y))$ is defined by

$$E^+(H; \mathcal{L}(Y)) := \sup \left\{ \int_0^1 H(t, \gamma(t)) dt \mid \gamma \in \mathcal{L}(Y) \right\}.$$

This is a stronger invariant in that $E^+(H; \mathcal{L}(Y)) \leq E^+(H; Y)$ and more directly related to the loop space $\mathcal{L}(Y)$ of Y . This formula suggests that we may use symplectic homology $SH(V)$ ([FH]) of a subset $V \subset M$ and the Viterbo functoriality (Viterbo [Vi2], Abouzaid-Seidel [ASe]) in place of Lagrangian Floer theory in certain cases, for example, in the case where V is a Darboux-Weinstein neighborhood of a Lagrangian submanifold L . For the case where the Floer homology $HF(L)$ is isomorphic to $H(L)$ (such as the case L is exact), the symplectic homology $SH(V)$ is related to the homology of the loop space of L (Salamon-Weber [SW], Abbondandolo-Schwartz [ASc1], which is in turn closely related to the Hochschild cohomology of $H(L)$. (See also [Fu2]).) Thus in that case the method using symplectic homology becomes equivalent to those using Hochschild cohomology that we have established in this section.

Eliashberg-Polterovich [EIP] use symplectic homology to estimate the spectral invariant in the case of Lagrangian tori in $S^2 \times S^2$. Through the above mentioned equivalence, their argument is related to ours given in Section 23.

Part 6. Appendix

26. $\mathcal{P}_{(H_\chi, J_\chi),*}^b$ IS AN ISOMORPHISM

In Section 6 we introduced the Piunikhin map $\mathcal{P}_{(H_\chi, J_\chi),*}^b$ with bulk deformation. In this section we complete the proof of Theorem 6.9:

Theorem 26.1. *The Piunikhin map with bulk deformation*

$$\mathcal{P}_{(H_\chi, J_\chi),*}^b : H_*(M; \Lambda^\downarrow) \rightarrow HF_*(M, H, J; \Lambda^\downarrow)$$

is a Λ^\downarrow -module isomorphism.

Proof. We first construct another map

$$\mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}}),*}^b : HF_*(M, H; \Lambda^\downarrow) \rightarrow H_*(M; \Lambda^\downarrow) \quad (26.1)$$

in the direction opposite to $\mathcal{P}_{(H_\chi, J_\chi),*}^b$. This will be carried out by constructing the associated chain map

$$CF(M, H; \Lambda^\downarrow) \rightarrow \Omega(M) \hat{\otimes} \Lambda^\downarrow. \quad (26.2)$$

Let $\chi \in \mathcal{K}$ be as in Definition 3.11 and $[\gamma, w] \in \text{Crit}(\mathcal{A}_H)$. For the construction of this chain map, we need to consider the dual version of χ . To distinguish the two different types of elongation functions, we recall that we denote

$$\tilde{\chi}(\tau) = -\chi(-\tau)$$

for $\chi \in \mathcal{K}$. We also use (H_χ, J_χ) defined in (3.12). (In this section $J = \{J_t\}$ is a $t \in S^1$ parametrized family of compatible almost complex structures.)

We consider the elongated family $(H_{\tilde{\chi}}, J_{\tilde{\chi}})$ defined by:

$$H_{\tilde{\chi}}(\tau, t, x) = \tilde{\chi}(\tau)H_t(x), \quad J_{\tilde{\chi}}(\tau, t) = J_{\tilde{\chi}(\tau), t},$$

where $J_{s,t}$ is as in (3.11).

Definition 26.2. We denote by $\mathring{\mathcal{M}}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ the set of all pairs

$$(u; z_1^+, \dots, z_\ell^+)$$

of maps $u : \mathbb{R} \times S^1 \rightarrow M$ and $z_i^+ \in \mathbb{R} \times S^1$ which satisfy the following conditions:

(1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J_{\tilde{\chi}} \left(\frac{\partial u}{\partial t} - X_{H_{\tilde{\chi}}}(u) \right) = 0. \quad (26.3)$$

(2) The energy

$$E_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}(u) = \frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_{\tilde{\chi}}}^2 + \left| \frac{\partial u}{\partial t} - X_{H_{\tilde{\chi}}}(u) \right|_{J_{\tilde{\chi}}}^2 \right) dt d\tau$$

is finite.

(3) The map u satisfies the following asymptotic boundary condition.

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = \gamma(t).$$

(4) The homology class of the concatenation of u and w is equivalent to 0 by the equivalence relation \sim .

(5) z_i^+ are all distinct each other.

$(u; z_1^+, \dots, z_\ell^+) \mapsto (u(z_1^+), \dots, u(z_\ell^+))$ defines an evaluation map

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) = \mathring{\mathcal{M}}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *) \rightarrow M^\ell.$$

Lemma 26.3. (1) *The moduli space $\mathring{\mathcal{M}}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ has a compactification*

$$\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$$

that is Hausdorff.

(2) *The space $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ has an orientable Kuranishi structure with corners.*

(3) *The boundary of $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ is described by*

$$\begin{aligned} & \partial \mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *) \\ &= \bigcup \mathcal{M}(H, J; [\gamma, w], [\gamma', w']) \times \mathcal{M}(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma', w'], *), \end{aligned} \quad (26.4)$$

where the union is taken over all $[\gamma', w'] \in \text{Crit}(H)$ and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

(4) *Let $\mu_H : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$, be the Conley-Zehnder index. Then the (virtual) dimension satisfies the following equality:*

$$\dim \mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *) = n - \mu_H([\gamma, w]) + 2\ell. \quad (26.5)$$

(5) *We can define orientations of $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ so that (3) above is compatible with this orientation.*

(6) *ev extends to a strongly continuous smooth map $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *) \rightarrow M^\ell$, which we denote also by ev . It is compatible with (3).*

(7) *The map $\text{ev}_{+\infty}$ which sends $(u; z_1^+, \dots, z_\ell^+)$ to $\lim_{\tau \rightarrow +\infty} u(\tau, t)$ extends to a weakly submersive map $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *) \rightarrow M$, which we also denote by $\text{ev}_{+\infty}$. It is compatible with (3).*

The proof of Lemma 26.3 is the same as that of Proposition 3.6 and so is omitted.

We take a system of continuous families of multisections $\{\mathbf{s}^w\}_{w \in W}$ on the moduli space $\mathcal{M}_\ell(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w], *)$ which is compatible with (3) and such that $\text{ev}_{+\infty}$ is a submersion on its zero set.

Let $h_1, \dots, h_\ell \in \Omega(M)$. We define $\mathbf{n}_{(H, J), *([\gamma, w])}(h_1, \dots, h_\ell) \in \Omega(M)$ by

$$\mathbf{n}_{(H, J), *([\gamma, w])}(h_1, \dots, h_\ell) = \text{ev}_{+\infty, !}(\text{ev}_1^* h_1 \wedge \dots \wedge \text{ev}_\ell^* h_\ell \wedge \omega_W). \quad (26.6)$$

Here $\text{ev}_{+\infty, !}$ is the integration along fiber of the map $\text{ev}_{+\infty}$ on the zero set of our family of multisections, and ω_W is a smooth form of top degree on the parameter space W such that $\int_W \omega_W = 1$. (See [FOOO3] Section 12.)

Let $\mathbf{b} \in H^{\text{even}}(M; \Lambda_0^\perp)$. We split $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_2 + \mathbf{b}_+$ as in (5.5). We take closed forms which represent \mathbf{b}_0 , \mathbf{b}_2 , \mathbf{b}_+ and write them by the same symbols.

Definition 26.4.

$$\begin{aligned} & \mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathbf{b}}([\gamma, w]) \\ &= \sum_{w'} \sum_{\ell=0}^{\infty} \frac{\exp(\int (w')^* \mathbf{b}_2)}{\ell!} q^{-\int w^* \omega + \int (w')^* \omega} \mathbf{n}_{(H, J), *([\gamma, w'])}(\underbrace{\mathbf{b}_+, \dots, \mathbf{b}_+}_{\ell}). \end{aligned} \quad (26.7)$$

We can prove that the sum in (26.7) converges in q -adic topology in the same way as in Lemma 6.5. We have thus defined (26.1). Then

$$\partial \circ \mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathbf{b}} = \mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathbf{b}} \circ \partial_{(H, J)}^{\mathbf{b}} \quad (26.8)$$

is a consequence of Lemma 26.3 (3) and Stokes' theorem. (Here ∂ is defined by (3.19).)

Proposition 26.5. $\mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^b \circ \mathcal{P}_{(H_{\chi}, J_{\chi})}^b$ is chain homotopic to the identity.

Proof. For $S \in [1, \infty)$ define H_{χ}^S as follows:

$$H_{\chi}^S(\tau, t, x) = \begin{cases} \chi(\tau + S + 1)H_t(x) & S \geq 1, \tau \leq 0 \\ \tilde{\chi}(\tau - S - 1)H_t(x) & S \geq 1, \tau \geq 0. \end{cases} \quad (26.9)$$

We extend it to $S \in [0, 1]$ by

$$H_{\chi}^S(\tau, t, x) = SH_{\chi}^1(\tau, t, x). \quad (26.10)$$

The function H_{χ}^S may not be smooth on S at $S = 1$, $\tau \in [-10, 10]$. We modify it on a neighborhood of $S = 1$, $\tau \in [-10, 10]$ so that it becomes smooth and denote it by the same symbol. We define $(S, \tau, t) \in [0, \infty) \times \mathbb{R} \times [0, 1]$ parametrized family of compatible almost complex structures J_{χ}^S as follows. For $S \in [1, \infty)$ we put

$$J_{\chi}^S(\tau, t) = \begin{cases} J_{\chi(\tau+S+1), t} & S \geq 1, \tau \leq 0, \\ J_{\tilde{\chi}(\tau-S-1), t} & S \geq 1, \tau \geq 0. \end{cases} \quad (26.11)$$

We extend it to $S \in [0, 1]$ so that the following is satisfied.

$$J_{\chi}^S(\tau, t) = \begin{cases} J_0 & \tau \leq -10, \\ J_0 & \tau \geq +10, \\ J_0 & S = 0, \\ J_0 & t \text{ is in a neighborhood of } [1]. \end{cases} \quad (26.12)$$

Definition 26.6. Let $C \in H_2(M; \mathbb{Z})$. For each $0 \leq S < \infty$, we denote by $\mathring{\mathcal{M}}_{\ell}(H_{\chi}^S, J_{\chi}^S; *, *; C)$ the set of all pairs $(u; z_1^+, \dots, z_{\ell}^+)$ of maps $u : \mathbb{R} \times S^1 \rightarrow M$, $z_i^+ \in \mathbb{R} \times S^1$ which satisfy the following conditions:

- (1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J_{\chi}^S \left(\frac{\partial u}{\partial t} - X_{H_{\chi}^S}(u) \right) = 0. \quad (26.13)$$

- (2) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_{\chi}^S}^2 + \left| \frac{\partial u}{\partial t} - X_{H_{\chi}^S}(u) \right|_{J_{\chi}^S}^2 \right) dt d\tau$$

is finite.

- (3) The homology class of u is C .

- (4) z_i^+ are all distinct each other.

We note that (26.13) and the finiteness of energy imply that there exist $p_1, p_2 \in M$ such that

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = p_1, \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = p_2. \quad (26.14)$$

Therefore the homology class of u is well-defined. We define the evaluation map

$$(\text{ev}_{-\infty}, \text{ev}_{+\infty}) : \mathring{\mathcal{M}}_{\ell}(H_{\chi}^S, J_{\chi}^S; *, *; C) \rightarrow M^2$$

by $(\text{ev}_{-\infty}, \text{ev}_{+\infty})(u) = (p_1, p_2)$, where p_1, p_2 are as in (26.14).

We put

$$\mathring{\mathcal{M}}_\ell(para; H_\chi, J_\chi; *, *, C) = \bigcup_{S \geq 0} \{S\} \times \mathring{\mathcal{M}}_\ell(H_\chi^S, J_\chi^S; *, *, C), \quad (26.15)$$

where ev , $\text{ev}_{-\infty}$ and $\text{ev}_{+\infty}$ are defined on it.

To describe the boundary of the compactification of $\mathring{\mathcal{M}}_\ell(para; H_\chi, J_\chi; *, *, C)$ we define another moduli space.

Definition 26.7. We denote by $\widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ the set of all $(u; z_1^+, \dots, z_\ell^+)$ that satisfy (1), ..., (4) of Definition 26.6 with $S = 0$.

Note that H actually does not appear in (1), ..., (4) of Definition 26.6 in case $S = 0$. There exists an $\mathbb{R} \times S^1$ action on $\widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ that is induced by the $\mathbb{R} \times S^1$ action on $\mathbb{R} \times S^1$, the source of the map u . In fact, the equation (26.13) is prepreserved by $\mathbb{R} \times S^1$ action in case $S = 0$.

We define evaluation maps

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) \rightarrow M^\ell$$

and

$$(\text{ev}_{+\infty}, \text{ev}_{-\infty}) : \widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) \rightarrow M^2$$

in an obvious way. We put

$$\begin{aligned} \mathring{\mathcal{M}}_\ell(H = 0, J_0; *, *, C) &= \widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) / \mathbb{R}, \\ \overline{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) &= \widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) / S^1, \\ \overline{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) &= \widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C) / (\mathbb{R} \times S^1). \end{aligned}$$

Then $\widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$, $\mathring{\mathcal{M}}_\ell(H = 0, J_0; *, *, C)$, $\overline{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ and $\overline{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ can be compactified. We denote the corresponding compactifications by $\widehat{\mathcal{M}}_\ell(H = 0, J_0; *, *, C)$, $\mathcal{M}_\ell(H = 0, J_0; *, *, C)$, $\widehat{\overline{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ and $\overline{\mathcal{M}}_\ell(H = 0, J_0; *, *, C)$, respectively. The compactifications are obtained as follows. Fix an identification of $\mathbb{R} \times S^1$ with $\mathbb{CP}^1 \setminus \{N, S\}$, where N, S are the limits as $\tau \rightarrow \pm\infty$, respectively. For each $(u; z_1^+, \dots, z_\ell^+) \in \widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$, we regard u as a map from \mathbb{CP}^1 and consider its graph in $\mathbb{CP}^1 \times M$. Then we identify the space $\widehat{\mathring{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ with the space $\mathring{\mathcal{N}}_\ell(H = 0, J_0; *, *, C)$ of their graphs. Take its stable map compactification $\mathcal{N}_\ell(H = 0, J_0; *, *, C)$, which is identified with $\widehat{\mathcal{M}}_\ell(H = 0, J_0; *, *, C)$. (The component, which has degree 1 to \mathbb{CP}^1 -factor is the component with a parametrized solution of (26.3).) The group $\mathbb{R} \times S^1$ acts on the first factor of $\mathbb{CP}^1 \times M$ and induces an action on $\mathcal{N}_\ell(H = 0, J_0; *, *, C)$. By taking the quotient of $\mathcal{N}_\ell(H = 0, J_0; *, *, C)$ by \mathbb{R} , S^1 , $\mathbb{R} \times S^1$, we obtain the compactification $\mathcal{M}_\ell(H = 0, J_0; *, *, C)$, $\widehat{\overline{\mathcal{M}}}_\ell(H = 0, J_0; *, *, C)$ and $\overline{\mathcal{M}}_\ell(H = 0, J_0; *, *, C)$, respectively. Each of them carries a Kuranishi structure and evaluation maps that

extend to its compactification. We note that $\overline{\mathcal{M}}_\ell(H = 0, J_0; *, *, C)$ is identified with $\mathcal{M}_{\ell+2}^{\text{cl}}(C)$ which is introduced in Section 5.

Lemma 26.8. (1) *The moduli space $\mathring{\mathcal{M}}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$ has a compactification $\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$ that is Hausdorff.*

(2) *The space $\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$ has an orientable Kuranishi structure with corners.*

(3) *The boundary of $\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$ is described by the union of following four types of direct or fiber products:*

(i)

$$\mathcal{M}_{\#\mathbb{L}_1}(H_\chi, J_\chi; *, [\gamma, w]) \times \mathcal{M}_{\#\mathbb{L}_2}(H_{\tilde{\chi}}, J_{\tilde{\chi}}; [\gamma, w'], *) \quad (26.16)$$

where the union is taken over all $[\gamma, w] \in \text{Crit}(H)$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$. Here the bounding disc w' is defined by $[w]\#C = [w']$.

(ii)

$$\mathcal{M}_{\#\mathbb{L}_1}(H = 0, J_0; *, *, C_1)_{\text{ev}_{+\infty}} \times_{\text{ev}_{-\infty}} \mathcal{M}_{\#\mathbb{L}_2}(\text{para}; H_\chi, J_\chi; *, *, C_2) \quad (26.17)$$

where the union is taken over all C_1, C_2 and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$ such that $C_1 + C_2 = C$. The fiber product is taken over M .

(iii)

$$\mathcal{M}_{\#\mathbb{L}_1}(\text{para}; H_\chi, J_\chi; *, *, C_1)_{\text{ev}_{+\infty}} \times_{\text{ev}_{-\infty}} \mathcal{M}_{\#\mathbb{L}_2}(H = 0, J_0; *, *, C_2) \quad (26.18)$$

where the union is taken over all C_1, C_2 and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$ such that $C_1 + C_2 = C$. The fiber product is taken over M .

(iv) And

$$\widehat{\mathcal{M}}_{\#\mathbb{L}}(H = 0, J_0; *, *, C). \quad (26.19)$$

(4) *The (virtual) dimension satisfies the following equality:*

$$\dim \mathcal{M}_\ell(\text{para}; H^\chi, J^\chi; *, *, C) = 2c_1(M) \cap C + 2n + 2\ell - 1. \quad (26.20)$$

(5) *We can define orientations of $\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$ so that (3) above is compatible with this orientation.*

(6) *ev extends to a weakly submersive map $\mathcal{M}_\ell(\text{para}; H_\chi, J_\chi; *, *, C) \rightarrow M^\ell$, which we denote also by ev. It is compatible with (3).*

(7) *(ev_{−∞}, ev₊∞) extends also to a weakly submersive map*

$$\mathcal{M}_\ell(H_\chi, J_\chi; *, *, C) \rightarrow M^2,$$

which we denote by (ev_{−∞}, ev₊∞). It is compatible with (3).

Proof. The proof is similar to the proof of Proposition 6.11. So we only mention the way how the four types of boundary components appear. In fact, (26.16) appears when $S \rightarrow \infty$, (26.19) appears when $S = 0$. (26.17), (26.18) appear when S is bounded and is away from 0. (26.17) is the case there is some bubble which slides to $\tau \rightarrow -\infty$ and (26.18) is the case there is some bubble which slides to $\tau \rightarrow +\infty$. \square

We now take a system of continuous families of multisections on $\mathcal{M}_\ell(H_\chi, J_\chi; *, *, C)$ such that it is compatible with the description of its boundary Lemma 26.8 (3) and that ev₊∞ is a submersion on the zero set of the continuous families of multisections. We need some particular choice of it at some of the factors of the boundary component.

We observe that there exist maps

$$\widehat{\mathcal{M}}_\ell(H = 0, J_0; *, *, C) \rightarrow \overline{\mathcal{M}}_\ell(H = 0, J_0; *, *, C) \quad (26.21)$$

and

$$\mathcal{M}_\ell(H = 0, J_0; *, *, C) \rightarrow \overline{\mathcal{M}}_\ell(H = 0, J_0; *, *, C). \quad (26.22)$$

Various evaluation maps factor through them. We take our family of multisections so that it is obtained by the pull back with respect to the maps (26.21), (26.22).

We use the family of multisections as above to define

$$\mathfrak{H}_{H,J;C}^b : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow \Omega(M) \widehat{\otimes} \Lambda^\downarrow$$

by

$$\mathfrak{H}_{H,J}^b(h) = \sum_{\ell=0}^{\infty} \sum_C \frac{\exp(C \cap \mathfrak{b}_2)}{\ell!} q^{-C \cap \omega} \text{ev}_{+\infty}! \left(\text{ev}_{-\infty}^* h \wedge \text{ev}^* (\underbrace{\mathfrak{b}_+, \dots, \mathfrak{b}_+}_{\ell}) \right).$$

Here each term of the right hand side is the correspondence by the moduli space $\mathring{\mathcal{M}}_\ell(\text{para}; H_\chi, J_\chi; *, *, C)$.

Lemma 26.9.

$$\partial \circ \mathfrak{H}_{H,J}^b + \mathfrak{H}_{H,J}^b \circ \partial = \mathcal{Q}_{(H_{\bar{\chi}}, J_{\bar{\chi}})}^b \circ \mathcal{P}_{(H_\chi, J_\chi)}^b - id.$$

Proof. The proof is based on Lemma 26.8 (3) and Stokes' theorem ([FOOO3] Lemma 12.13). We note that (26.16) corresponds to the composition $\mathcal{Q}_{(H_{\bar{\chi}}, J_{\bar{\chi}})}^b \circ \mathcal{P}_{(H_\chi, J_\chi)}^b$. Using the compatibility of the multisection and evaluation map to (26.21), (26.22) it is easy to see that the contribution of (26.17) and (26.18) vanishes.

By the same reason the contribution of (26.19) vanishes except the case $\ell = 0$ and $C = 0$. In that case the moduli space is M and $\text{ev}_{\pm\infty}$ is the identity map. Therefore the contribution is the identity map : $\Omega(M) \rightarrow \Omega(M)$. This finishes the proof of Lemma 26.9. \square

Therefore the proof of Proposition 26.5 is now complete. \square

In a similar way as in Proposition 26.5 we can prove that $\mathcal{P}_{(H_\chi, J_\chi)}^b \circ \mathcal{Q}_{(H_{\bar{\chi}}, J_{\bar{\chi}})}^b$ is chain homotopic to identity. Hence the proof of Theorem 26.1 is now complete. \square

We now complete the proof of Theorem 7.8 (3). It remains to prove the following:

Proposition 26.10.

$$\rho^b(\underline{Q}; a) \geq \mathfrak{v}_q(a).$$

Proof.

Lemma 26.11. *If $\mathcal{M}_\ell(H_{\bar{\chi}}, J_{\bar{\chi}}; [\gamma, w], *)$ is nonempty, then*

$$\mathcal{A}_H([\gamma, w]) \geq -E^+(H).$$

The proof is similar to the proof of Lemma 9.8 and so is omitted.

Corollary 26.12.

$$\mathcal{Q}_{(H_{\bar{\chi}}, J_{\bar{\chi}})}^b (F^\lambda CF(M, H, J; \Lambda^\downarrow)) \subseteq q^{\lambda + E^+(H)} \Omega(M) \widehat{\otimes} \Lambda^\downarrow.$$

Proof. Let $x \in F^\lambda CF(M, H, J; \Lambda^\downarrow)$. We choose w_γ for each of $\gamma \in \text{Per}(H)$ and put

$$x = \sum_{\gamma \in \text{Per}(H)} x_\gamma [\gamma, w_\gamma],$$

with

$$\mathfrak{v}_q(x_\gamma) + \mathcal{A}_H([\gamma, w_\gamma]) \leq \lambda. \quad (26.23)$$

By (26.7) we have

$$\mathfrak{v}_q(\mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathfrak{b}}(x)) \leq \max_{[\gamma, w'_\gamma]} (-w_\gamma \cap \omega + w'_\gamma \cap \omega + \mathfrak{v}_q(x_\gamma)), \quad (26.24)$$

where the maximum in the right hand side is taken over all $[\gamma, w'_\gamma] \in \text{Crit}(\mathcal{A}_H)$ such that $\mathcal{M}_\ell(H, J; [\gamma, w'_\gamma], *)$ is nonempty.

We note that

$$-w_\gamma \cap \omega + w'_\gamma \cap \omega = -\mathcal{A}_H([\gamma, w'_\gamma]) + \mathcal{A}_H([\gamma, w_\gamma]). \quad (26.25)$$

By (26.23), (26.24), (26.25) we obtain

$$\mathfrak{v}_q(\mathcal{Q}_{(H_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathfrak{b}}(x)) \leq \lambda + E^+(H)$$

as required. \square

We take a sequence of normalized Hamiltonians H_i such that $\lim_{i \rightarrow \infty} \|H_i\| = 0$ and $\tilde{\psi}_{H_i}$ is non-degenerate. Let $x \in CF(M, H_i, J; \Lambda^\downarrow)$ such that $\partial_{(H_i, J)}^{\mathfrak{b}} x = 0$, $[x] = \mathcal{P}_{((H_i)_\chi, J_\chi)}^{\mathfrak{b}}(a^{\mathfrak{b}})$, and

$$|\mathfrak{v}_q(x) - \rho^{\mathfrak{b}}(H_i; a)| < \epsilon.$$

Then $[\mathcal{Q}_{((H_i)_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathfrak{b}}(x)] = a$ and

$$\mathfrak{v}_q(\mathcal{Q}_{((H_i)_{\tilde{\chi}}, J_{\tilde{\chi}})}^{\mathfrak{b}}(x)) \leq \rho^{\mathfrak{b}}(H_i; a) + \epsilon + E^+(H_i).$$

Since ϵ is arbitrary small and $\lim_{i \rightarrow \infty} E^+(H_i) = 0$, we obtain the proposition. \square

27. INDEPENDENCE OF DE RHAM REPRESENTATIVE OF \mathfrak{b} .

In this section we prove Theorem 7.7 (2). Let H be a one periodic Hamiltonian on M such that ψ_H is nondegenerate. Let $\mathfrak{b}(0), \mathfrak{b}(1) \in \Omega(M) \hat{\otimes} \Lambda^\downarrow$ such that $d\mathfrak{b}(0) = d\mathfrak{b}(1) = 0$. We assume that there exists $\mathfrak{c} \in \Omega(M) \hat{\otimes} \Lambda^\downarrow$ such that

$$\mathfrak{b}(1) - \mathfrak{b}(0) = d\mathfrak{c}. \quad (27.1)$$

Then we prove that $\rho^{\mathfrak{b}(0)}(\tilde{\phi}_H, a) = \rho^{\mathfrak{b}(1)}(\tilde{\phi}_H, a)$. Firstly we consider the case that $\mathfrak{b}_2(0) = \mathfrak{b}_2(1)$. Here $\mathfrak{b}_2(0), \mathfrak{b}_2(1) \in H^2(M; \mathbb{C})$ as in (5.5). After establishing Theorem 7.7 (2) under the condition that $\mathfrak{b}_2(0) = \mathfrak{b}_2(1)$, we show that the invariant $\rho^{\mathfrak{b}}(\tilde{\phi}_H, a)$ does not depend on the choice of representative of the cohomology class $[\mathfrak{b}_2]$.

We consider the ring of strongly convergent power series

$$\Lambda^\downarrow \langle\langle s \rangle\rangle = \left\{ \sum_{k=0}^{\infty} x_k s^k \mid x_k \in \Lambda^\downarrow, \lim_{k \rightarrow \infty} \mathfrak{v}_q(x_k) = -\infty \right\}. \quad (27.2)$$

Here s is a formal parameter. We denote by $\text{Poly}(\mathbb{R}; CF(M; H; \Lambda^\downarrow))$ the set of formal expressions of the form

$$x(s) + ds \wedge y(s)$$

where

$$x(s), y(s) \in CF(M; H; \Lambda^\downarrow) \otimes_{\Lambda^\downarrow} \Lambda^\downarrow \langle\langle s \rangle\rangle.$$

For $s_0 \in \mathbb{R}$ we define

$$\text{Eval}_{s=s_0} : \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^\downarrow)) \rightarrow CF(M; H; \Lambda^\downarrow)$$

by

$$\text{Eval}_{s=s_0}(x(s) + ds \wedge y(s)) = x(s_0). \quad (27.3)$$

We note that, for $x(s) = \sum_{k=0}^{\infty} x_k s^k \in CF(M; H; \Lambda^\downarrow) \otimes_{\Lambda^\downarrow} \Lambda^\downarrow \langle\langle s \rangle\rangle$ with $x_k \in CF(M; H; \Lambda^\downarrow)$, the series $x(s_0) = \sum_{k=0}^{\infty} x_k s_0^k$ converges in q -adic topology for $s_0 \in \mathbb{R}$.

We put

$$\mathfrak{b}(s) = s\mathfrak{b}(1) + (1-s)\mathfrak{b}(0). \quad (27.4)$$

For each $s_0 \in \mathbb{R}$ we define

$$\partial_{(H,J)}^{\mathfrak{b}(s_0)} : CF(M; H; \Lambda^\downarrow) \rightarrow CF(M; H; \Lambda^\downarrow)$$

by (6.6).

Lemma 27.1. *There exists a $\Lambda^\downarrow \langle\langle s \rangle\rangle$ -module homomorphism*

$$\partial_{(H,J)}^{\mathfrak{b}(\cdot)} : CF(M; H; \Lambda^\downarrow) \otimes_{\Lambda^\downarrow} \Lambda^\downarrow \langle\langle s \rangle\rangle \rightarrow CF(M; H; \Lambda^\downarrow) \otimes_{\Lambda^\downarrow} \Lambda^\downarrow \langle\langle s \rangle\rangle$$

such that

$$\text{Eval}_{s=s_0} \circ \partial_{(H,J)}^{\mathfrak{b}(\cdot)} = \partial_{(H,J)}^{\mathfrak{b}(s_0)} \circ \text{Eval}_{s=s_0}, \quad \partial_{(H,J)}^{\mathfrak{b}(\cdot)} \circ \partial_{(H,J)}^{\mathfrak{b}(\cdot)} = 0. \quad (27.5)$$

Proof. We split $\mathfrak{b}(s) = \mathfrak{b}_0(s) + \mathfrak{b}_2(s) + \mathfrak{b}_+(s)$ as in (5.5). Then we have $\mathfrak{b}_2(s) = s\mathfrak{b}_2(1) + (1-s)\mathfrak{b}_2(0)$ etc. We use it to see that

$$\mathfrak{n}_{(H,J);\ell}([\gamma, w], [\gamma', w']) \underbrace{(\mathfrak{b}_+(s), \dots, \mathfrak{b}_+(s))}_{\ell}$$

is a polynomial of order $\leq \ell$ in s with coefficients in \mathbb{C} . (See (6.4), (6.5).)

By (6.5), we find that

$$\mathfrak{n}_{(H,J)}^{\mathfrak{b}(s)}([\gamma, w], [\gamma', w']) \in \Lambda^\downarrow \langle\langle s \rangle\rangle.$$

Hence we can define $\partial_{(H,J)}^{\mathfrak{b}(\cdot)}$ by replacing \mathfrak{b} by $\mathfrak{b}(s)$ in (6.6). The first formula in (27.5) is easy to show. The second formula follows from the first one. \square

We next put

$$\begin{aligned} & \mathfrak{n}_{(H,J)}^{\mathfrak{c},1}([\gamma, w], [\gamma', w']) \\ &= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \frac{\exp(w' \cap \mathfrak{b}_2(s) - w \cap \mathfrak{b}_2(s))}{(\ell_1 + \ell_2 + 1)!} \\ & \quad \mathfrak{n}_{(H,J);\ell_1+\ell_2+1}([\gamma, w], [\gamma', w']) \underbrace{(\mathfrak{b}_+(s), \dots, \mathfrak{b}_+(s))}_{\ell_1} \underbrace{(\mathfrak{c}, \mathfrak{b}_+(s), \dots, \mathfrak{b}_+(s))}_{\ell_2} \\ & \in \Lambda^\downarrow \langle\langle s \rangle\rangle \end{aligned}$$

and define

$$\partial_{(H,J)}^{\mathfrak{c}}([\gamma, w]) = \sum_{[\gamma', w']} \mathfrak{n}_{(H,J)}^{\mathfrak{c},1}([\gamma, w], [\gamma', w']) [\gamma', w']. \quad (27.6)$$

Lemma 27.2.

$$\frac{\partial}{\partial s} \circ \partial_{(H,J)}^{\mathbf{b}(\cdot)} - \partial_{(H,J)}^{\mathbf{b}(\cdot)} \circ \frac{\partial}{\partial s} = \partial_{(H,J)}^{\epsilon} \circ \partial_{(H,J)}^{\mathbf{b}(\cdot)} - \partial_{(H,J)}^{\mathbf{b}(\cdot)} \circ \partial_{(H,J)}^{\epsilon}. \quad (27.7)$$

Proof. Using Proposition 6.2 (3) and Stokes' formula, we obtain

$$\begin{aligned} & \sum_{i=1}^{\ell} (-1)^* \mathbf{n}_{(H,J);\ell}([\gamma, w], [\gamma', w'])(h_1, \dots, dh_i, \dots, h_{\ell}) \\ &= \sum_{(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)} \sum_{[\gamma'', w'']} (-1)^{**} \mathbf{n}_{(H,J);\#\mathbb{L}_1}([\gamma, w], [\gamma'', w''])(h_{i_1}, \dots, h_{i_{\#\mathbb{L}_1}}) \\ & \quad \mathbf{n}_{(H,J);\#\mathbb{L}_2}([\gamma'', w''], [\gamma', w'])(h_{j_1}, \dots, h_{j_{\#\mathbb{L}_2}}), \end{aligned} \quad (27.8)$$

where $\mathbb{L}_1 = \{i_1, \dots, i_{\#\mathbb{L}_1}\}$, $\mathbb{L}_2 = \{j_1, \dots, j_{\#\mathbb{L}_2}\}$,

$$* = \deg h_1 + \dots + \deg h_{i-1}, \quad ** = \sum_{i \in \mathbb{L}_1, j \in \mathbb{L}_2; j < i} \deg h_i \deg h_j.$$

Using (27.4) and (27.8) we can prove Lemma 27.2 easily. \square

We define

$$\partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)} : \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow})) \rightarrow \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow}))$$

by

$$\begin{aligned} & \partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)}(x(s) + ds \wedge y(s)) \\ &= \partial_{(H,J)}^{\mathbf{b}(\cdot)}(x(s)) - ds \wedge \frac{\partial}{\partial s}(x(s)) + ds \wedge \partial_{(H,J)}^{\epsilon}(x(s)) - ds \wedge \partial_{(H,J)}^{\mathbf{b}(\cdot)}(y(s)). \end{aligned} \quad (27.9)$$

Then the second formula of (27.5) and Lemma 27.2 imply

$$\partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)} \circ \partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)} = 0.$$

Thus $(\text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow})), \partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)})$ is a chain complex. The first formula of (27.5) implies that

$$\text{Eval}_{s=s_0} : (\text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow})), \partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)}) \rightarrow (CF(M; H; \Lambda^{\downarrow}), \partial_{(H,J)}^{\mathbf{b}(s_0)}) \quad (27.10)$$

is a chain map.

We define a filtration $F^{\lambda} \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow}))$ by

$$\begin{aligned} & F^{\lambda} \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow})) \\ &= \{x(s) + ds \wedge y(s) \mid x(s) = \sum x_k s^k, x(s) = \sum y_k s^k, \mathbf{v}_q(x_k), \mathbf{v}_q(y_k) \leq \lambda\}. \end{aligned}$$

Lemma 27.3. $\partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)}$ and $\text{Eval}_{s=s_0}$ preserves the filtration F^{λ} .

The proof is easy and is omitted.

Lemma 27.4. The map (27.10) is a chain homotopy equivalence.

Proof. If $x(s) + ds \wedge y(s) \in F^{\lambda} \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow}))$, then we have

$$\partial_{(H,J)}^{(\mathbf{b}(\cdot), \epsilon)}(x(s) + ds \wedge y(s)) - ds \wedge \frac{\partial x}{\partial s}(s) \in F^{\lambda-\epsilon} \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^{\downarrow}))$$

for some positive ϵ . We use this fact to prove Lemma 27.4 in the same way as in the proof of [FOOO1] Proposition 4.3.18. \square

We next define

$$\mathcal{P}_{(H_X, J_X)}^{(\mathfrak{b}(\cdot), \mathfrak{c})} : \Omega(M) \otimes \Lambda^\downarrow \rightarrow \text{Poly}(\mathbb{R}; CF(M; H; \Lambda^\downarrow)). \quad (27.11)$$

For each fixed s_0 we define

$$\mathcal{P}_{(H_X, J_X)}^{\mathfrak{b}(s_0)} : \Omega(M) \otimes \Lambda^\downarrow \rightarrow CF(M; H; \Lambda^\downarrow) \quad (27.12)$$

by (6.12). We then obtain

$$\mathcal{P}_{(H_X, J_X)}^{\mathfrak{b}(\cdot)} : \Omega(M) \otimes \Lambda^\downarrow \rightarrow CF(M; H; \Lambda^\downarrow) \otimes_{\Lambda^\downarrow} \Lambda^\downarrow \langle\langle s \rangle\rangle \quad (27.13)$$

such that

$$\text{Eval}_{s=s_0} \circ \mathcal{P}_{(H_X, J_X)}^{\mathfrak{b}(\cdot)} = \mathcal{P}_{(H_X, J_X)}^{\mathfrak{b}(s_0)}. \quad (27.14)$$

Let

$$\begin{aligned} & \mathfrak{n}_{(H_X, J_X)}^{\mathfrak{c}, 1}(h; [\gamma, w]) \\ &= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \frac{\exp(\int w^* \mathfrak{b}_2(s))}{(\ell_1 + \ell_2 + 1)!} \\ & \quad \mathfrak{n}_{(H, J); \ell_1 + \ell_2 + 1}([\gamma, w]) \underbrace{(\mathfrak{b}_+(s), \dots, \mathfrak{b}_+(s))}_{\ell_1} \underbrace{(\mathfrak{c}, \mathfrak{b}_+(s), \dots, \mathfrak{b}_+(s))}_{\ell_2} \\ & \in \Lambda^\downarrow \langle\langle s \rangle\rangle. \end{aligned}$$

(See (6.11).) We use this to define

$$\mathcal{P}_{(H_X, J_X)}^{\mathfrak{c}}(h) = \sum_{[\gamma, w]} \mathfrak{n}_{(H_X, J_X)}^{\mathfrak{c}, 1}(h; [\gamma, w]) [\gamma, w].$$

Now we put

$$\mathcal{P}_{(H_X, J_X)}^{(\mathfrak{b}(\cdot), \mathfrak{c})}(h) = \mathcal{P}_{(H_X, J_X)}^{\mathfrak{b}(\cdot)}(h) + ds \wedge \mathcal{P}_{(H_X, J_X)}^{\mathfrak{c}}(h). \quad (27.15)$$

Lemma 27.5. *We have*

$$\partial_{(H, J)}^{(\mathfrak{b}(\cdot), \mathfrak{c})} \circ \mathcal{P}_{(H_X, J_X)}^{(\mathfrak{b}(\cdot), \mathfrak{c})} = \mathcal{P}_{(H_X, J_X)}^{(\mathfrak{b}(\cdot), \mathfrak{c})} \circ \partial$$

and

$$\text{Eval}_{s=s_0} \circ \partial_{(H, J)}^{(\mathfrak{b}(\cdot), \mathfrak{c})} = \mathcal{P}_{(H_X, J_X)}^{\mathfrak{b}(s_0)}.$$

The proof is straightforward calculation and is omitted.

We use Lemmas 27.3, 27.4, 27.5 and can prove $\rho^{\mathfrak{b}(0)}(\tilde{\psi}_H, a) = \rho^{\mathfrak{b}(1)}(\tilde{\psi}_H, a)$ easily. The proof of Theorem 7.7 (2) is complete under the condition that $\mathfrak{b}_2(0) = \mathfrak{b}_2(1)$. \square

Next, for $\mathfrak{b}(0), \mathfrak{b}(1)$ such that $\mathfrak{b}(1) - \mathfrak{b}(0) = d\mathfrak{c}$ for some \mathfrak{c} , we consider $\mathfrak{b}' = \mathfrak{b}(0) + d(\mathfrak{c} - \mathfrak{c}_1)$. Here \mathfrak{c}_1 is the $\Omega^1(M; \mathbb{C})$ -component of \mathfrak{c} in the decomposition $\Omega^1(M; \mathbb{C}) \oplus \Omega^1(M; \Lambda^\downarrow) \oplus \Omega^{\geq 3}(M; \Lambda^\downarrow)$. We showed that $\rho^{\mathfrak{b}(0)}(\tilde{\phi}_H, a) = \rho^{\mathfrak{b}'}(\tilde{\phi}_H, a)$. The remaining task is to show that $\rho^{\mathfrak{b}'}(\tilde{\phi}_H, a) = \rho^{\mathfrak{b}(1)}(\tilde{\phi}_H, a)$. Namely we prove Theorem 7.7 (2) in the case that $\mathfrak{b}(1) - \mathfrak{b}(0) = d\mathfrak{c}_1$ with $\mathfrak{c}_1 \in \Omega^1(M; \mathbb{C})$.

We define $I : CF(M; H; \Lambda^\downarrow) \rightarrow CF(M; H; \Lambda^\downarrow)$ by

$$I([\gamma, w]) = \exp\left(\int_{S^1} \gamma^* \mathfrak{c}_1\right) [\gamma, w].$$

Then we find that I gives an isomorphism of Floer chain complexes

$$I : (CF(M; H; \Lambda^\downarrow), \partial_{(J, H)}^{\mathfrak{b}'}) \rightarrow (CF(M; H; \Lambda^\downarrow), \partial_{(J, H)}^{\mathfrak{b}(1)})$$

and

$$I \circ \mathcal{P}_{(H_X, J_X)}^{\mathbf{b}'} = \mathcal{P}_{(H_X, J_X)}^{\mathbf{b}(1)}.$$

Hence the proof of Theorem 7.7 (2). \square

Remark 27.6. A cocycle $\mathbf{b} \in \Omega^2(M; \mathbb{C})$ induces a representation

$$\mathbf{rep}_{\mathbf{b}} : a \in \pi_1(\mathcal{L}(M); \ell_0) \mapsto \exp\left(\int_{C_a} \mathbf{b}\right) \in \mathbb{C}^*,$$

where $C_a : S^1 \times S^1 \rightarrow M$ is the mapping corresponding to the loop a in $\mathcal{L}(M)$. Then we can consider Floer complex of the Hamiltonian system with coefficients in the local system corresponding to $\mathbf{rep}_{\mathbf{b}}$. If $\mathbf{b}(0)$ and $\mathbf{b}(1)$ are cohomologous, the corresponding local systems are isomorphic, hence Floer cohomology with coefficients in these local systems are isomorphic. Here we gave the isomorphism I directly without dealing with the isomorphism of the local systems.

28. PROOF OF PROPOSITION 20.6.

The purpose of this section is to prove Proposition 20.6 and Lemma 20.8. In this section we fix t -independent J .

28.1. Pseudo-isotopy of filtered A_∞ algebra. In this subsection, we review the notion of pseudo-isotopy of filtered A_∞ algebra, which was introduced in [Fu3] Definition 8.5. We consider $L \times [0, 1]$ and use s for the coordinate of $[0, 1]$. We put $\overline{C} = \Omega(L)$ and

$$C^\infty([0, 1] \times \overline{C}) = \Omega([0, 1] \times L).$$

An element of $C^\infty([0, 1] \times \overline{C})$ is written uniquely as

$$x(s) + ds \wedge y(s)$$

where $x(s), y(s)$ are smooth differential forms on $[0, 1] \times L$ that do not contain ds . For each fixed s_0 we have $x(s_0), y(s_0) \in \overline{C}$.

Suppose that, for each $s \in [0, 1]$, $k, \ell, \beta \in \pi_2(M; L)$ we have operators

$$\mathbf{m}_{k, \beta}^s : B_k(\overline{C}[1]) \rightarrow \overline{C}[1] \quad (28.1)$$

of degree $-\mu(\beta) + 1$ and

$$\mathbf{c}_{k, \beta}^s : B_k(\overline{C}[1]) \rightarrow \overline{C}[1] \quad (28.2)$$

of degree $-\mu(\beta)$.

Definition 28.1. We say $\mathbf{m}_{k, \beta}^s$ is *smooth* if for each $x_1, \dots, x_k \in \overline{C}$ we may regard

$$\mathbf{m}_{k, \beta}^s(x_1, \dots, x_k)$$

as an element of $C^\infty([0, 1], \overline{C})$ without ds component. The smoothness of $\mathbf{c}_{k, \beta}^s$ is defined in the same way.

Suppose that there exists a subset \widehat{G} of $H_2(M, L; \mathbb{Z})$ such that $\{\omega \cap \beta \mid \beta \in \widehat{G}\}$ is a discrete subset of $\mathbb{R}_{\geq 0}$. Let G be the monoid generated by this set. We assume that we have $\mathbf{m}_{k, \beta}^s, \mathbf{c}_{k, \beta}^s$ for $\beta \in \widehat{G}$ only.

Definition 28.2. We say $(C, \{\mathbf{m}_{k, \beta}^s\}, \{\mathbf{c}_{k, \beta}^s\})$ is a *pseudo-isotopy* of G -gapped filtered A_∞ algebras if the following holds:

- (1) $\mathbf{m}_{k, \beta}^s$ and $\mathbf{c}_{k, \beta}^s$ are smooth.
- (2) For each (but fixed) s , $(C, \{\mathbf{m}_{k, \beta}^s\})$ defines a filtered A_∞ algebra.

(3) For each $x_i \in \overline{C}[1]$

$$\begin{aligned} & \frac{d}{ds} \mathbf{m}_{k,\beta}^s(x_1, \dots, x_k) \\ & + \sum_{k_1+k_2=k} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} (-1)^* \mathbf{c}_{k_1,\beta_1}^s(x_1, \dots, \mathbf{m}_{k_2,\beta_2}^s(x_i, \dots), \dots, x_k) \\ & - \sum_{k_1+k_2=k} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} \mathbf{m}_{k_1,\beta_1}^s(x_1, \dots, \mathbf{c}_{k_2,\beta_2}^s(x_i, \dots), \dots, x_k) \\ & = 0. \end{aligned} \quad (28.3)$$

Here $*$ = $\deg' x_1 + \dots + \deg' x_{i-1}$.

(4) \mathbf{m}_{k,β_0}^s is independent of s , and $\mathbf{c}_{k,\beta_0}^s = 0$. Here $\beta_0 = 0 \in H_2(M; L; \mathbb{Z})$.

We consider $x_i(s) + ds \wedge y_i(s) = \mathbf{x}_i \in C^\infty([0, 1], \overline{C})$. We define

$$\widehat{\mathbf{m}}_{k,\beta}(\mathbf{x}_1, \dots, \mathbf{x}_k) = x(s) + ds \wedge y(s), \quad (28.4)$$

where

$$x(s) = \mathbf{m}_{k,\beta}^s(x_1(s), \dots, x_k(s)) \quad (28.5a)$$

$$\begin{aligned} y(s) = & \mathbf{c}_{k,\beta}^s(x_1(s), \dots, x_k(s)) \\ & - \sum_{i=1}^k (-1)^{*i} \mathbf{m}_{k,\beta}^t(x_1(s), \dots, x_{i-1}(s), y_i(s), x_{i+1}(s), \dots, x_k(s)) \end{aligned} \quad (28.5b)$$

if $(k, \beta) \neq (1, \beta_0)$ and

$$y(s) = \frac{d}{ds} x_1(s) + \mathbf{m}_{1,0}^s(y_1(s)) \quad (28.5c)$$

if $(k, \beta) = (1, \beta_0)$. Here $*i$ in (28.5b) is $*i = \deg' x_1 + \dots + \deg' x_{i-1}$.

Lemma 28.3. *The equation (28.3) is equivalent to the filtered A_∞ relation of $\widehat{\mathbf{m}}_{k,\beta}$ defined by (28.5).*

The proof is a straightforward calculation.

Definition 28.4. A pseudo-isotopy $(C, \{\mathbf{m}_{k,\beta}^s\}, \{\mathbf{c}_{k,\beta}^s\})$ is said to be *unital* if there exists $\mathbf{e} \in \overline{C}^0$ such that \mathbf{e} is a unit of $(C, \{\mathbf{m}_{k,\beta}^s\})$ for each s and if

$$\mathbf{c}_{k,\beta}^s(\dots, \mathbf{e}, \dots) = 0$$

for each k, β and s .

In our situation the unit \mathbf{e} is always \mathbf{e}^L , the constant function 1 on L .

Theorem 28.5. *If $(C, \{\mathbf{m}_{k,\beta}^s\}, \{\mathbf{c}_{k,\beta}^s\})$ is a unital pseudo-isotopy, then there exists a unital filtered A_∞ homomorphism from $(C, \{\mathbf{m}_{k,\beta}^0\})$ to $(C, \{\mathbf{m}_{k,\beta}^1\})$ that has a homotopy inverse.*

Proof. The cyclic version of this theorem is [Fu3] Theorem 8.2. Since we do not require cyclic symmetry here, the proof of Theorem 28.5 is easier. In fact, it follows from [FOOO1] Theorem 4.2.45 as follows. We have a filtered A_∞ homomorphism

$$\text{Eval}_{s=s_0} : (C^\infty([0, 1] \times \overline{C}), \{\widehat{\mathbf{m}}_{k,\beta}^s\}) \rightarrow (\overline{C}, \{\mathbf{m}_{k,\beta}^{s_0}\})$$

defined by

$$(\text{Eval}_{s=s_0})_1(a(s) + ds \wedge b(s)) = a(s_0)$$

and

$$(\text{Eval}_{s=s_0})_k = 0$$

for $k \neq 1$. Then using [FOOO1] Theorem 4.2.45 we can show that it is a homotopy equivalence. Theorem 28.5 follows. \square

28.2. Difference between \mathfrak{m}^T and \mathfrak{m} . We will construct a pseudo-isotopy between two filtered A_∞ structures $\{\mathfrak{m}_{k,\beta}^{T,\mathfrak{b}}\}$ and $\{\mathfrak{m}_{k,\beta}^{\mathfrak{b}}\}$ on $\overline{\mathcal{C}} = \Omega(L)$. Here the first one is defined in Section 20 and the second one is defined in Section 17. We note that the difference of these two constructions are roughly as follows:

- (1) We represent \mathfrak{b} by a T^n invariant cycle D_a that is a submanifold to define $\{\mathfrak{m}_{k,\beta}^{T,\mathfrak{b}}\}$. In other words, in the definition of $\{\mathfrak{m}_{k,\beta}^{T,\mathfrak{b}}\}$, we use that current which may not be smooth. On the other hand, we represent \mathfrak{b} by a smooth differential forms to define $\{\mathfrak{m}_{k,\beta}^{\mathfrak{b}}\}$.
- (2) In the definition of $\{\mathfrak{m}_{k,\beta}^{T,\mathfrak{b}}\}$ we first take the fiber product (20.9) and then use a multisection to achieve transversality. On the other hand, to define $\{\mathfrak{m}_{k,\beta}^{\mathfrak{b}}\}$, we first perturb (by a family of multisections) the moduli space $\mathcal{M}_{k+1;\ell}(\beta)$ then pull back the differential form representing \mathfrak{b} to the zero set of the multisection. In other words the perturbation to define $\{\mathfrak{m}_{k,\beta}^{\mathfrak{b}}\}$ is independent of the ambient cohomology class \mathfrak{b} .

Remark 28.6. We note that there are various reasons why, when we construct $\{\mathfrak{m}_{k,\beta}^{T,\mathfrak{b}}\}$ in the toric case, we need to take cycles and multisections (rather than taking a family of multisections). The most important reason is Proposition 20.10. This is related to point (1) above. The reason why we first need to take the fiber product (20.9) is explained in [FOOO2] Remark 11.4.

On the other hand to develop the theory of spectral invariant with bulk deformation in the general setting, it seems simplest to always use de Rham representative.

We will construct a pseudo-isotopy of filtered A_∞ structures interpolating $\{\mathfrak{m}_{k,\beta}^{T,\mathfrak{b}}\}$ and $\{\mathfrak{m}_{k,\beta}^{\mathfrak{b}}\}$. Below we handle the above (1) and (2) separately. We construct the pseudo-isotopy resolving (1) in Subsection 28.3 and construct the pseudo-isotopy resolving (2) in Subsection 28.5.

28.3. Smoothing T^n -invariant chains. Let $D_a = D_{i_1} \cap \cdots \cap D_{i_k}$ be a transversal intersection of k irreducible components of the toric divisor, ND_a its normal bundle, and $\exp : ND_a \rightarrow M$ the exponential map with respect to a T^n -invariant Riemannian metric. Let $\mathcal{U}_a \subset \Gamma(ND_a)$ be a finite dimensional submanifold of the space of smooth sections of ND_a such that if $\mathbf{u} \in \mathcal{U}_a$ and $\rho \in [0, 1]$ then $\rho\mathbf{u} \in \mathcal{U}_a$. We assume that it has the following properties.

Properties 28.7. (1) The exponential map $\text{Exp} : D_a \times \mathcal{U}_a \rightarrow M$ defined by

$$\text{Exp}(\mathbf{u}, x) = \exp(\mathbf{u}(x)) \quad (28.6)$$

is a submersion.

- (2) $\|\mathbf{u}(x)\| < \epsilon$, where ϵ is a sufficiently small positive number determined later.

We put $d_a = \dim \mathcal{U}_a$.

Let $\mathbf{p} : \{1, \dots, \ell\} \rightarrow \underline{B}$ be as in the beginning of Subsection 20.2. We put

$$\mathcal{U}(\mathbf{p}) = \prod_{i=1}^{\ell} \mathcal{U}_{\mathbf{p}(i)}, \quad \mathbf{p}(\mathcal{U}) = \prod_{i=1}^{\ell} (D_{\mathbf{p}(i)} \times \mathcal{U}_{\mathbf{p}(i)}). \quad (28.7)$$

The map (28.6) induces

$$\text{Exp} : \mathbf{p}(\mathcal{U}) \rightarrow M^\ell. \quad (28.8)$$

For $k, \ell \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(M, L(\mathbf{u}); \mathbb{Z})$ we define a fiber product

$$\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) = \mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta)_{(ev_1, \dots, ev_\ell)} \times_{\text{Exp}} \mathbf{p}(\mathcal{U}), \quad (28.9)$$

where $\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta)$ is a moduli space defined in Definition 17.2 and Proposition 17.3. (Compare (20.9).) We can define an evaluation map at the boundary marked points:

$$\text{ev}^\partial = (\text{ev}_1^\partial, \dots, \text{ev}_\ell^\partial) : \mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \rightarrow L(\mathbf{u})^{k+1}$$

in an obvious way. We also have a projection

$$\pi_{\mathcal{U}} : \mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \rightarrow \mathcal{U}(\mathbf{p})$$

to the \mathcal{U}_a -factors. By definition we have

$$\pi_{\mathcal{U}}^{-1}(\mathbf{0}) = \mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}). \quad (28.10)$$

Lemma 28.8. (1) $\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ has a Kuranishi structure with corners.

(2) It coincides with the Kuranishi structure in Lemma 20.4 on $\pi_{\mathcal{U}}^{-1}(\mathbf{0})$.

(3) Its boundary is described by the union of fiber products:

$$\mathcal{M}_{k_1+1; \# \mathbb{L}_1}(L(\mathbf{u}); \beta_1; \mathbf{p}_1(\mathcal{U}))_{\text{ev}_0^\partial} \times_{\text{ev}_i^\partial} \mathcal{M}_{k_2+1; \# \mathbb{L}_2}(L(\mathbf{u}); \beta_2; \mathbf{p}_2(\mathcal{U})) \quad (28.11)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$, k_1, k_2 with $k_1 + k_2 = k$ and $\beta_1, \beta_2 \in H_2(M, L(\mathbf{u}); \mathbb{Z})$ with $\beta = \beta_1 + \beta_2$. We put $\text{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p}) = (\mathbf{p}_1, \mathbf{p}_2)$.

(4) The dimension is

$$\begin{aligned} & \dim \mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \\ &= n + \mu_{L(\mathbf{u})}(\beta) + k - 2 + 2\ell - \sum_{i=1}^{\ell} 2 \deg D_{\mathbf{p}(i)} + \sum_{i=1}^{\ell} d_{\mathbf{p}(i)}. \end{aligned} \quad (28.12)$$

(5) The evaluation maps ev_i^∂ at the boundary marked points of $\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta)$ define maps on $\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$, which we denote by ev_i^∂ also. They are compatible with (3).

(6) We can define an orientation of the Kuranishi structure so that it is compatible with (3).

(7) $\text{ev}_0^\partial \times \pi_{\mathcal{U}}$ is weakly submersive.

(8) The Kuranishi structure is compatible with the action of the symmetry group \mathfrak{S}_ℓ .

(9) The Kuranishi structure is compatible with the forgetful map of the 1st, 2nd, ..., k -th boundary marked points. (We do not require that it is compatible with the forgetful map of the 0-th marked point.)

The proof is the same as that of Lemma 20.4. We note that (7) is a consequence of (2) if we take ϵ in Properties 28.7 (2) to be small enough.

Lemma 28.9. There exists a system of multisections on $\mathcal{M}_{k+1; \ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ with the following properties.

(1) They are transversal to $\mathbf{0}$.

(2) They coincide with the multisection in Lemma 20.5 on $\pi_{\mathcal{U}}^{-1}(\mathbf{0})$.

- (3) They are compatible with the description of the boundary in Lemma 20.4 (3).
- (4) The restriction of $\text{ev}_0^\partial \times \pi_{\mathcal{U}}$ to the zero set of this multisection is a submersion.
- (5) They are invariant under the action of \mathfrak{S}_ℓ .
- (6) They are compatible with the forgetful map of the 1st, 2nd, \dots , k -th boundary marked points.

The proof is mostly the same as the proof of Lemma 20.5. We only observe that (4) is a consequence of (2) if ϵ is sufficiently small.

For each $a = 1, \dots, B$ we choose a compactly supported smooth differential form χ_a of top degree on \mathcal{U}_a such that $\int_{\mathcal{U}_a} \chi_a = 1$. For $\mathbf{p} : \{1, \dots, \ell\} \rightarrow \underline{B}$ we put

$$\chi_{\mathbf{p}} = \prod_{i=1}^{\ell} \chi_{\mathbf{p}(i)} \in \Omega(\mathcal{U}(\mathbf{p})).$$

Let $h_1, \dots, h_k \in \Omega(L(\mathbf{u}))$. We then define a differential form on $L(\mathbf{u})$ by

$$\mathfrak{q}_{\ell,k;\beta}^S(\mathbf{p}; h_1, \dots, h_k) = (\text{ev}_0^\partial)_! (\text{ev}_1^\partial, \dots, \text{ev}_k^\partial, \pi_{\mathcal{U}})^*(h_1 \wedge \dots \wedge h_k \wedge \chi_{\mathbf{p}}), \quad (28.13)$$

where we use the evaluation map

$$(\text{ev}_0^\partial, \dots, \text{ev}_k^\partial, \pi_{\mathcal{U}}) : \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \rightarrow L(\mathbf{u})^{k+1} \times \mathcal{U}(\mathbf{p})$$

and $(\text{ev}_0)_!$ is the integration along the fiber. Here the superscript S stands for smoothing. By Lemma 28.9 (4) integration along the fiber is well-defined. By Lemma 28.9 (5) the operators $\mathfrak{q}_{\ell,k;\beta}^S$ is invariant under the permutation of components of \mathbf{p} . Therefore by the \mathbb{C} -linearity we define

$$\mathfrak{q}_{\ell,k;\beta}^S : E_\ell(\mathcal{H}[2]) \otimes B_k(\Omega(L(\mathbf{u}))[1]) \rightarrow \Omega(L(\mathbf{u}))[1]. \quad (28.14)$$

We use it in the same way as in Definition 17.7 to define $\mathfrak{m}_k^{S,\mathbf{b}}$ for $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_{2;1}, \mathbf{b}_+, b_+)$. Thus we have obtained a filtered A_∞ algebra $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{S,\mathbf{b}}\}_{k=0}^\infty)$. Here we recall

$$CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0) = \Omega(L(\mathbf{u})) \hat{\otimes} \Lambda_0.$$

Lemma 28.10. *The filtered A_∞ algebra $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{S,\mathbf{b}}\}_{k=0}^\infty)$ is pseudo-isotopic to $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{T,\mathbf{b}}\}_{k=0}^\infty)$ as a unital filtered A_∞ algebra.*

Proof. Let δ_0^a be the distributional d_a form on \mathcal{U}_a supported at $\mathbf{0}$ and satisfy $\int \delta_0^a = 1$. (Namely it is the delta function times the volume form.) We also take a distributional $d_a - 1$ form κ_a on \mathcal{U}_a with the following properties.

- Properties 28.11.** (1) $d\kappa_a = \chi_a - \delta_0^a$.
 (2) κ_a is smooth outside the origin.

We put

$$\chi_a^s = s\chi_a + (1-s)\delta_0^a \quad (28.15)$$

and

$$\chi_{\mathbf{p}}^s = \prod_{i=1}^{\ell} \chi_{\mathbf{p}(i)}^s$$

that is a distributional $\sum d_{\mathbf{p}(i)}$ form on $\mathcal{U}(\mathbf{p})$. We then define

$$\mathfrak{q}_{\ell,k;\beta}^{S,s}(\mathbf{p}; h_1, \dots, h_k) = (\text{ev}_0^\partial)_! (\text{ev}_1^\partial, \dots, \text{ev}_k^\partial)^*(h_1 \wedge \dots \wedge h_k \wedge \chi_{\mathbf{p}}^s). \quad (28.16)$$

Note that $\chi_{\mathbf{p}}^s$ is a distributional form so the existence of pull back is not automatic. However we can show that the pull-back exists and the right hand side of (28.16) is a smooth differential form by Lemma 28.9 (4).

The map $\mathfrak{q}_{\ell,k;\beta}^{S,s}$ induces

$$\mathfrak{q}_{\ell,k;\beta}^{S,s} : E_{\ell}(\mathcal{H}[2]) \otimes B_k(\Omega(L(\mathbf{u}))[1]) \rightarrow \Omega(L(\mathbf{u}))[1]. \quad (28.17)$$

We use it to define $\mathfrak{m}_k^{S,s,\mathbf{b}}$ in the same way as in Definition 17.7. Then it is smooth (with respect to s coordinate) in the sense of Definition 28.4.

Sublemma 28.12. *$(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{S,s,\mathbf{b}}\}_{k=0}^{\infty})$ is a unital filtered A_{∞} algebra. Moreover we have:*

$$\mathfrak{m}_k^{S,0,\mathbf{b}} = \mathfrak{m}_k^{T,\mathbf{b}}, \quad \mathfrak{m}_k^{S,1,\mathbf{b}} = \mathfrak{m}_k^{S,\mathbf{b}}.$$

The proof is easy and omitted.

We next denote

$$\kappa_{i,\mathbf{p}}^s = \chi_{\mathbf{p}(1)}^s \wedge \cdots \wedge \chi_{\mathbf{p}(i-1)}^s \wedge \kappa_{\mathbf{p}(i)}^s \wedge \chi_{\mathbf{p}(i+1)}^s \wedge \cdots \wedge \chi_{\mathbf{p}(\ell)}^s$$

and define

$$\begin{aligned} \mathfrak{qc}_{\beta;\ell,k}^{S,s}(\mathbf{p}; h_1, \dots, h_k) \\ = \sum_{i=1}^{\ell} (-1)^{*(i)} (\text{ev}_0^{\partial})! (\text{ev}_1^{\partial}, \dots, \text{ev}_k^{\partial})^* (h_1 \wedge \cdots \wedge h_k \wedge \kappa_{i,\mathbf{p}}^s). \end{aligned} \quad (28.18)$$

See Remark 28.16 for the sign. In the same way as the operator $\mathfrak{q}_{\ell,k;\beta}^{S,s}$ defines $\mathfrak{m}_k^{S,s,\mathbf{b}}$, the operator $\mathfrak{qc}_{\beta;\ell,k}^{S,s}$ induces an operator, which we write $\mathfrak{c}_k^{S,s,\mathbf{b}}$. It is easy to see that

$$(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{S,s,\mathbf{b}}\}_{k=0}^{\infty}, \{\mathfrak{c}_k^{S,s,\mathbf{b}}\}_{k=0}^{\infty})$$

is the required pseudo-isotopy. The proof of Lemma 28.10 is complete. \square

28.4. Completion of the proof of Proposition 20.6. In this subsection, we construct a pseudo-isotopy between $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{S,\mathbf{b}}\}_{k=0}^{\infty})$ (which is defined in Subsection 28.3) and $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{\mathbf{b}}\}_{k=0}^{\infty})$ (which is defined in Definition 17.7.)

Together with Theorem 28.5 and Lemma 28.10, this will complete the proof of Proposition 20.6.

In Section 27, we already proved that the homotopy equivalence class of

$$(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathfrak{m}_k^{\mathbf{b}}\}_{k=0}^{\infty})$$

is independent of the choice of de Rham representative \mathbf{b} . We make this choice more specifically below.

Let D_a be as in the beginning of Subsection 28.3. We put

$$\mathfrak{b}_a = \text{Exp}_!(\pi_{\mathcal{U}}^* \chi_a), \quad (28.19)$$

where we use $(\text{Exp}, \pi_{\mathcal{U}}) : D_a \times \mathcal{U}_a \rightarrow M \times \mathcal{U}_a$. Clearly \mathfrak{b}_a is a de Rham representative of the Poincaré dual to $[D_a]$. The de Rham cohomology classes $\{[\mathfrak{b}_a]\}_{a=1}^B$ form a basis of $\bigoplus_{k \neq 0} H^k(M; \mathbb{C})$. We use them to specify the de Rham representatives of the elements of $\bigoplus_{k \neq 0} H^k(M; \Lambda)$. (We represent the 0-th cohomology class by the constant function.)

We next review two Kuranishi structures and two families of multisections on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$.

(Kuranishi structure and multisections 1) Consider the natural projection

$$\pi : \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \rightarrow \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta). \quad (28.20)$$

We have chosen and fixed a Kuranishi structure on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta)$ in Proposition 17.3. We pull it back by the map (28.20). It defines a Kuranishi structure \mathcal{K}_1 .

In Lemma 17.4 we took and fixed a continuous family of multisections on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta)$. We pull it back by the map (28.20) and obtain a continuous family of multisections of the Kuranishi structure \mathcal{K}_1 . We denote it by $\mathfrak{s}_1 = \{\mathfrak{s}_{1,w}\}_{w \in W}$. We also took a top degree differential form of compact support χ_W on W satisfying $\int \chi_W = 1$. We use them to define $\mathfrak{q}_{\ell,k;\beta}^{\mathfrak{s}_1}$ by

$$\mathfrak{q}_{\ell,k;\beta}^{\mathfrak{s}_1}(\mathbf{p}; h_1, \dots, h_k) = (\mathrm{ev}_0^\partial)_!((\mathrm{ev}_1^\partial, \dots, \mathrm{ev}_k^\partial, \pi_{\mathcal{U}})^*(h_1 \wedge \dots \wedge h_k \wedge \chi_{\mathbf{p}}) \wedge \chi_W), \quad (28.21)$$

where we use the evaluation map

$$(\mathrm{ev}_0^\partial, \dots, \mathrm{ev}_k^\partial, \pi_{\mathcal{U}}) : \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))^{\mathfrak{s}_1} \rightarrow L(\mathbf{u})^{k+1} \times \mathcal{U}(\mathbf{p}),$$

from the zero set of the family of multisections \mathfrak{s}_1 .

Lemma 28.13. $\mathfrak{q}_{\ell,k;\beta}^{\mathfrak{s}_1} = \mathfrak{q}_{\ell,k;\beta}$, where the right hand side is (17.9).

This lemma is obvious from the definition and (28.19).

(Kuranishi structure and multisections 2) In Lemma 28.8, we took a Kuranishi structure on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$. We call it the Kuranishi structure \mathcal{K}_2 . In Lemma 28.9, we took a multisection of \mathcal{K}_2 . We call it the multisection \mathfrak{s}_2 . They determine the operators $\mathfrak{q}_{\ell,k;\beta}^{\mathfrak{s}_2}$ by (28.13).

Thus we have described two systems of Kuranishi structures and multisections. We next define a system of Kuranishi structures and multisections on $[0, 1] \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ which interpolate them.

We define

$$\widehat{\mathrm{ev}}_i^\partial : [0, 1] \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \rightarrow [0, 1] \times L(\mathbf{u})$$

by $\widehat{\mathrm{ev}}_i^\partial = (\pi_s, \mathrm{ev}_i^\partial)$ where π_s is the projection to $[0, 1]$ factor. (We use s as the coordinate of this factor.)

Lemma 28.14. (1) $[0, 1] \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ has a Kuranishi structure with corners.

- (2) Its restriction to $\{0\} \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ coincides with \mathcal{K}_1 and its restriction to $\{1\} \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ coincides with \mathcal{K}_2 .
- (3) Its boundary is described by the union of

$$\partial([0, 1]) \times \mathcal{M}_{k_1+1;\ell_1}(L(\mathbf{u}); \beta_1; \mathbf{p}_1(\mathcal{U})).$$

and the union of fiber products

$$\begin{aligned} & \left([0, 1] \times \mathcal{M}_{k_1+1;\ell_1}(L(\mathbf{u}); \beta_1; \mathbf{p}_1(\mathcal{U})) \right) \\ & \widehat{\mathrm{ev}}_0^\partial \times_{\widehat{\mathrm{ev}}_i^\partial} \left([0, 1] \times \mathcal{M}_{k_2+1;\ell_2}(L(\mathbf{u}); \beta_2; \mathbf{p}_2(\mathcal{U})) \right) \end{aligned} \quad (28.22)$$

where the union is taken over all $(\mathbb{L}_1, \mathbb{L}_2) \in \mathrm{Shuff}(\ell)$, k_1, k_2 with $k_1 + k_2 = k$ and $\beta_1, \beta_2 \in H_2(M, L(\mathbf{u}); \mathbb{Z})$ with $\beta = \beta_1 + \beta_2$. We put $\mathrm{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p}) = (\mathbf{p}_1, \mathbf{p}_2)$.

(4) *The dimension is*

$$\begin{aligned} & \dim \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \\ &= n + \mu_{L(\mathbf{u})}(\beta) + k - 1 + 2\ell - \sum_{i=1}^{\ell} 2 \deg D_{\mathbf{p}(i)} + \sum_{i=1}^{\ell} d_{\mathbf{p}(i)}. \end{aligned} \quad (28.23)$$

- (5) *The evaluation maps $\widehat{\text{ev}}_i^{\partial}$ at the boundary marked points of $\mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta)$ define a map on $\mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$, which we denote by $\widehat{\text{ev}}_i^{\partial}$ also. It is compatible with (3).*
- (6) *We can define an orientation of the Kuranishi structure so that it is compatible with (3).*
- (7) *$\widehat{\text{ev}}_0^{\partial} \times \pi_{\mathcal{U}}$ is weakly submersive.*
- (8) *The Kuranishi structure is compatible with the action of the symmetry group \mathfrak{S}_{ℓ} .*
- (9) *The Kuranishi structure is compatible with the forgetful map of the 1st, 2nd, ..., k -th boundary marked points. (We do not require that it is compatible with the forgetful map of the 0-th marked point.)*

The proof is the same as in Lemma 20.4 and is omitted.

Lemma 28.15. *There exists a system of families of multisections of the Kuranishi structure on $[0, 1] \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ in Lemma 28.14 with the following properties.*

- (1) *They are transversal to 0.*
- (2) *Its restriction to $\{0\} \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ coincides with \mathfrak{s}_1 and its restriction to $\{1\} \times \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U}))$ coincides with \mathfrak{s}_2 .*
- (3) *They are compatible with the description of the boundary in Lemma 28.14 (3).*
- (4) *The restriction of $\widehat{\text{ev}}_0^{\partial} \times \pi_{\mathcal{U}}$ to the zero set of this multisection is a submersion.*
- (5) *They are invariant under the action of \mathfrak{S}_{ℓ} .*
- (6) *They are compatible with the forgetful map of the 1st, 2nd, ..., k -th boundary marked points.*

The proof is the same as the proof of Lemma 20.5 and is omitted.

We now define operators

$$\mathfrak{q}_{\ell,k;\beta}^{para} : E_{\ell}(\mathcal{H}[2]) \otimes B_k(\Omega(L(\mathbf{u}))[1]) \rightarrow \Omega([0, 1] \times L(\mathbf{u}))[1]$$

as follows.

$$\mathfrak{q}_{\ell,k;\beta}^{para}(\mathbf{p}; h_1, \dots, h_k) = (\widehat{\text{ev}}_0^{\partial})_!((\widehat{\text{ev}}_1^{\partial}, \dots, \widehat{\text{ev}}_k^{\partial}, \pi_{\mathcal{U}})^*(h_1 \wedge \dots \wedge h_k \wedge \chi_{\mathbf{p}}) \wedge \chi_W), \quad (28.24)$$

where we use the evaluation map

$$(\widehat{\text{ev}}_0^{\partial}, \dots, \widehat{\text{ev}}_k^{\partial}, \pi_{\mathcal{U}}) : \mathcal{M}_{k+1;\ell}(L(\mathbf{u}); \beta; \mathbf{p}(\mathcal{U})) \rightarrow ([0, 1] \times L(\mathbf{u}))^{k+1} \times \mathcal{U}(\mathbf{p})$$

and $(\widehat{\text{ev}}_0^{\partial})_!$ is the integration along the fiber.

We divide it into the sum of the form which does not contain ds and one which contains ds and write:

$$\mathfrak{q}_{\ell,k;\beta}^{para} = \mathfrak{q}_{\ell,k;\beta}^{para,1} + ds \wedge \mathfrak{q}_{\ell,k;\beta}^{para,2}.$$

Now we put

$$\begin{aligned} & \mathbf{m}_k^{\mathbf{b}}(x_1, \dots, x_k) \\ &= \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell=0}^{\infty} \sum_{m_0=0}^{\infty} \dots \sum_{m_k=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{\ell!} \\ & \quad \mathfrak{q}_{\ell, k + \sum_{i=0}^k m_i; \beta}^{para, 1}(\mathbf{b}_+^{\otimes \ell}; b_+^{\otimes m_0}, x_1, b_+^{\otimes m_1}, \dots, b_+^{\otimes m_{k-1}}, x_k, b_+^{\otimes m_k}), \end{aligned} \quad (28.25)$$

$$\begin{aligned} & \mathbf{c}_k^{\mathbf{b}}(x_1, \dots, x_k) \\ &= \sum_{\beta \in H_2(M, L; \mathbb{Z})} \sum_{\ell=0}^{\infty} \sum_{m_0=0}^{\infty} \dots \sum_{m_k=0}^{\infty} T^{\omega \cap \beta} \frac{\exp(\mathbf{b}_{2;1} \cap \beta)}{\ell!} \\ & \quad \mathfrak{q}_{\ell, k + \sum_{i=0}^k m_i; \beta}^{para, 2}(\mathbf{b}_+^{\otimes \ell}; b_+^{\otimes m_0}, x_1, b_+^{\otimes m_1}, \dots, b_+^{\otimes m_{k-1}}, x_k, b_+^{\otimes m_k}). \end{aligned} \quad (28.26)$$

They define maps from $B_k(\Omega(L(\mathbf{u})) \widehat{\otimes} \Lambda)$ to $(\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda)$. By Lemmas 28.14, 28.15, $\mathbf{m}_k^{\mathbf{b}}$ and $\mathbf{c}_k^{\mathbf{b}}$ define a unital pseudo-isotopy between $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathbf{m}_k^{S, \mathbf{b}}\}_{k=0}^{\infty})$ and $(CF_{\text{dR}}(L(\mathbf{u}); \Lambda_0), \{\mathbf{m}_k^{\mathbf{b}}\}_{k=0}^{\infty})$. The proof of Proposition 20.6 is now complete. \square

Remark 28.16. The way to handle the sign in the argument of this section is the same as in [FOOO2]. (See the end of Appendix C [FOOO2].)

28.5. Proof of Lemma 20.8. In this subsection we prove Lemma 20.8. Let

$$\mathbf{m}_k^{1, \mathbf{b}} : B_k((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda)[1]) \rightarrow (\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda)[1]$$

be the filtered A_{∞} structure induced from the pseudo-isotopy in the proof of Lemma 28.10. Let

$$\mathbf{m}_k^{2, \mathbf{b}} : B_k((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda)[1]) \rightarrow (\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda)[1]$$

be the filtered A_{∞} structure induced from the pseudo-isotopy in Subsection 28.4.

They induce chain complexes

$$((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{1, \mathbf{b}}), \quad ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_2^{1, \mathbf{b}}).$$

We have chain homotopy equivalences

$$\text{Eval}_{s=0} : ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{1, \mathbf{b}}) \rightarrow (\Omega(L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{T, \mathbf{b}}),$$

$$\text{Eval}_{s=1} : ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{1, \mathbf{b}}) \rightarrow (\Omega(L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{S, \mathbf{b}}),$$

and

$$\text{Eval}_{s=0} : ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{2, \mathbf{b}}) \rightarrow (\Omega(L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{S, \mathbf{b}}),$$

$$\text{Eval}_{s=1} : ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{2, \mathbf{b}}) \rightarrow (\Omega(L(\mathbf{u})) \widehat{\otimes} \Lambda), \mathbf{m}_1^{\mathbf{b}}),$$

that are defined by (27.3).

Therefore to prove Lemma 20.8 it suffices to construct chain maps:

$$i_{\text{qm}, \mathbf{b}}^S : \Omega(M) \widehat{\otimes} \Lambda \rightarrow \Omega(L(\mathbf{u})) \widehat{\otimes} \Lambda; \mathbf{m}_1^{T, \mathbf{b}},$$

$$i_{\text{qm}, \mathbf{b}}^1 : \Omega(M) \widehat{\otimes} \Lambda \rightarrow ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda); \mathbf{m}_1^{1, \mathbf{b}}),$$

$$i_{\text{qm}, \mathbf{b}}^2 : \Omega(M) \widehat{\otimes} \Lambda \rightarrow ((\Omega([0, 1] \times L(\mathbf{u})) \widehat{\otimes} \Lambda); \mathbf{m}_1^{2, \mathbf{b}}),$$

such that

$$\text{Eval}_{s=0} \circ i_{\text{qm}, \mathbf{b}}^1 = i_{\text{qm}, \mathbf{b}}^T, \quad \text{Eval}_{s=1} \circ i_{\text{qm}, \mathbf{b}}^1 = i_{\text{qm}, \mathbf{b}}^S,$$

$$\text{Eval}_{s=0} \circ i_{\text{qm}, \mathbf{b}}^2 = i_{\text{qm}, \mathbf{b}}^S, \quad \text{Eval}_{s=1} \circ i_{\text{qm}, \mathbf{b}}^2 = i_{\text{qm}, \mathbf{b}}.$$

We can construct such $i_{\text{qm}, \mathbf{b}}^S$, $i_{\text{qm}, \mathbf{b}}^1$, $i_{\text{qm}, \mathbf{b}}^2$ by modifying the definition of $i_{\text{qm}, \mathbf{b}}$ (17.17) in an obvious way. The proof of Lemma 20.8 is complete. \square

29. SEIDEL HOMOMORPHISM WITH BULK

In this section we generalize Seidel homomorphism [Se] to a version with bulk deformation. We then generalize, in the next section, the result by Entov-Polterovich [EP1] section 4 and McDuff-Tolman [MT] on the relationship between Seidel homomorphism and Calabi quasimorphism. These generalizations are rather straightforward and do not require novel ideas.

29.1. Seidel homomorphism with bulk. In this subsection, we present a version of Seidel's construction [Se] that incorporates bulk deformations.

Let H be a one-periodic Hamiltonian such that $\phi_H : [0, 1] \rightarrow \text{Ham}(M, \omega)$ defines a loop, i.e. satisfies $\psi_H = \text{id}$. Such a loop is called a *Hamiltonian loop*. For such H , there is a diffeomorphism $\text{Per}(H) \cong M$. We fix this diffeomorphism by putting

$$z_p^H(t) = \phi_H^t(p). \quad (29.1)$$

Then the map $p \mapsto z_p^H$ is a one-to-one correspondence $M \rightarrow \text{Per}(H)$.

Let $v : \mathbb{R} \times S^1 \rightarrow M$ be a continuous map. We define $u : \mathbb{R} \times S^1 \rightarrow M$ by

$$u(\tau, t) = \phi_H^t(v(\tau, t)). \quad (29.2)$$

Lemma 29.1. *Let $p_-, p_+ \in M$. Then*

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = z_{p_-}^H(t), \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = z_{p_+}^H(t),$$

if and only if

$$\lim_{\tau \rightarrow -\infty} v(\tau, t) = p_-, \quad \lim_{\tau \rightarrow +\infty} v(\tau, t) = p_+.$$

The proof is a straightforward calculation. For a map u satisfying the above conditions, we define $[u] \in H_2(M; \mathbb{Z})$ by $[u] = [v]$. (Note v extends to a map from S^2 so $[v] \in H_2(M; \mathbb{Z})$ is defined.)

We define a symplectic fibration

$$\pi : E_{\phi_H} \rightarrow \mathbb{C}P^1$$

with fiber isomorphic to (M, ω) as follows. Let D_{\pm} be two copies of the unit disc in \mathbb{C} . Set $U_1 = D_- \times M$, $U_2 = (\mathbb{R} \times S^1) \times M$ and $U_3 = D_+ \times M$. We glue them by the gluing maps

$$I_- : (-\infty, 0) \times S^1 \times M \rightarrow D_- \setminus \{0\} \times M, \quad I_-((\tau, t), x) = (e^{2\pi(\tau + \sqrt{-1}t)}, x)$$

(where we regard $S^1 = \mathbb{R}/\mathbb{Z}$) and

$$I_+ : (1, \infty) \times S^1 \times M \rightarrow D_+ \setminus \{\infty\} \times M, \quad I_+((\tau, t), x) = (e^{-2\pi(\tau - 1 + \sqrt{-1}t)}, (\phi_H^t)^{-1}(x)).$$

We thus obtain

$$E_{\phi_H} = U_1 \cup U_2 \cup U_3.$$

The projections to the second factor induce a map

$$\pi : E_{\phi_H} \rightarrow D_- \cup (\mathbb{R} \times S^1) \cup D_+ \cong \mathbb{C}P^1.$$

This defines a locally trivial fiber bundle and the fiber of π is diffeomorphic to M .

In fact, $E_{\phi_H} \rightarrow \mathbb{C}P^1$ becomes a Hamiltonian fiber bundle. See [GLS] for the precise definition of Hamiltonian fiber bundle and its associated coupling form Ω

that we use below. We also refer to [Sc2, E, Oh4] for their applications to the Floer theory and spectral invariants.

Lemma 29.2. *The fibration $E_{\phi_H} \rightarrow \mathbb{CP}^1$ is a Hamiltonian fiber bundle, i.e., it carries a coupling form Ω on E_{ϕ_H} such that*

- (1) Ω is closed and $\Omega|_{E_{\phi_H, \gamma}} = \omega$,
- (2) $\pi_1 \Omega^{n+1} = 0$ where π_1 is the integration over fiber and $2n = \dim M$.

Proof. On each of U_i , $i = 1, 2, 3$, we pull back ω by the projection to M and denote it by ω_i . We put $\omega'_2 = \omega_2 + d(\chi H dt)$. Then we find that ω_1 on U_1 , ω'_2 on U_2 and ω_3 on U_3 are glued to a closed 2-form Ω on E_{ϕ_H} . The normalization condition on H then gives rise to the condition $\pi_1 \Omega^{n+1} = 0$. \square

Let $u : \mathbb{R} \times S^1 \rightarrow M$ be a continuous map. We denote the associated section $\hat{u} : \mathbb{R} \times S^1 \rightarrow E_{\phi_H}$ by the formula

$$\hat{u}(\tau, t) = ((\tau, t), u(\tau, t)). \quad (29.3)$$

Lemma 29.3. *Let $u : \mathbb{R} \times S^1 \rightarrow M$ be a continuous map. The following is equivalent:*

- (1) *There exists some $p_-, p_+ \in M$ such that*

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = p_-, \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = z_{p_+}^H(t).$$

- (2) *The map \hat{u} extends to a section $s_u : \mathbb{CP}^1 \rightarrow E_{\phi_H}$.*

The proof is obvious by definition of E_{ϕ_H} . Let u_1, u_2 satisfy the condition (1) above. We say that u_1 is homologous to u_2 if

$$[\hat{u}_1] = [\hat{u}_2] \in H_2(E_{\phi_H}; \mathbb{Z}).$$

Let $\Pi_2(M; H)$ be the set of the homology classes of such u . We note that

$$[\hat{u}_1] - [\hat{u}_2] \in \text{Ker}(H_2(E_{\phi_H}; \mathbb{Z}) \rightarrow H_2(\mathbb{CP}^1; \mathbb{Z})).$$

Therefore $\Pi_2(M; H)$ is a principal homogeneous space of the group $\text{Ker}(H_2(E_{\phi_H}; \mathbb{Z}) \rightarrow H_2(\mathbb{CP}^1; \mathbb{Z}))$.

We also have a natural marking $M \cong E_{\{0\}}$ of the fibration $E_{\phi_H} \rightarrow \mathbb{CP}^1$ via the map

$$M \times \{0\} \subset M \times \mathbb{C} \subset E_{\phi_H}$$

which we will fix once and for all. Then the natural inclusion induces a map $H_2(M; \mathbb{Z}) \rightarrow \text{Ker}(H_2(E_{\phi_H}; \mathbb{Z}) \rightarrow H_2(\mathbb{CP}^1; \mathbb{Z}))$. Therefore there exists an action

$$H_2(M; \mathbb{Z}) \times \Pi_2(M; H) \rightarrow \Pi_2(M; H) \quad (29.4)$$

of the group $H_2(M; \mathbb{Z})$ to $\Pi_2(M; H)$.

Remark 29.4. Theorem 29.9 which we will prove later implies that

$$H_2(M; \mathbb{Q}) \cong \text{Ker}(H_2(E_{\phi_H}; \mathbb{Q}) \rightarrow H_2(\mathbb{CP}^1; \mathbb{Q})).$$

We however do not use this fact.

Let J_0 be a compatible almost complex structure on M . For $t \in S^1$, we define

$$J_t^H = (\phi_H^t)_* J_0. \quad (29.5)$$

Since ϕ_H^t is a symplectic diffeomorphism, J_t^H is compatible with ω . We denote by $J^H = \{J_t^H\}_{t \in S^1}$ the above S^1 -parametrized family of compatible almost complex structures.

We take $\chi \in \mathcal{K}$ and consider H_χ as in (3.12). We also take an $(\mathbb{R} \times S^1)$ -parametrized family of almost complex structures J_χ^H such that

$$J_\chi^H(\tau, t) = \begin{cases} J_0 & \tau \leq 0, \\ J_t^H & \tau \geq 1, \\ J_0 & t \text{ is in a neighborhood of } [0] \in S^1. \end{cases} \quad (29.6)$$

Definition 29.5. For $\alpha \in \Pi_2(M; H)$ we denote by $\mathring{\mathcal{M}}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \mathbb{R} \times S^1 \rightarrow M$ and $z_1^+, \dots, z_\ell^+ \in \mathbb{R} \times S^1$, which satisfy the following conditions:

- (1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J_\chi^H \left(\frac{\partial u}{\partial t} - \chi(\tau) X_{H_t}(u) \right) = 0. \quad (29.7)$$

- (2) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_\chi^H}^2 + \left| \frac{\partial u}{\partial t} - \chi(\tau) X_{H_t}(u) \right|_{J_\chi^H}^2 \right) dt d\tau$$

is finite.

- (3) The map u satisfies the condition that there exists $p_+ \in M$ such that

$$\lim_{\tau \rightarrow +\infty} u(\tau, t) = z_{p_+}^H(t).$$

- (4) The homology class of u in $\Pi_2(M; H)$ is α .

- (5) z_i^+ are all distinct.

By our construction, the map

$$\bar{u} : \mathbb{R} \times S^1 \rightarrow M, \quad \bar{u}(\tau, t) = (\phi_H^t)^{-1} u(\tau, t)$$

is J_0 -holomorphic on $[1, \infty) \times S^1$ on M . Therefore we can apply removable singularity theorem to \bar{u} which gives rise to a section \hat{u} mentioned in Lemma 29.3.

We denote by

$$\text{ev}_{\pm\infty} : \mathring{\mathcal{M}}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha) \rightarrow M$$

the map which associates to u the limit $\lim_{\tau \rightarrow \pm\infty} u(\tau, 0)$. We define the evaluation maps at z_i :

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathring{\mathcal{M}}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha) \rightarrow (E_{\phi_H})^\ell$$

by

$$\text{ev}_i(u; z_1, \dots, z_\ell) = (z_i, u(z_i)) \in U_2 \subset E_{\phi_H}.$$

Definition 29.6. For $\alpha \in H_2(M; \mathbb{Z})$ we define $\widehat{\mathring{\mathcal{M}}}_\ell(H, J^H; z_*^H, z_*^H; \alpha)$ as the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \mathbb{R} \times S^1 \rightarrow M$ and $z_1^+, \dots, z_\ell^+ \in \mathbb{R} \times S^1$, which satisfy the following conditions:

- (1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J^H \left(\frac{\partial u}{\partial t} - X_H(u) \right) = 0. \quad (29.8)$$

(2) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J^H}^2 + \left| \frac{\partial u}{\partial t} - X_H(u) \right|_{J^H}^2 \right) dt d\tau$$

is finite.

(3) There exist points $p_{\pm} \in M$ such that

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = z_{p_-}^H, \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = z_{p_+}^H.$$

(4) The homology class of u is α .

(5) z_i^+ are all distinct.

There is an \mathbb{R} -action on $\widehat{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ that is induced by the translation of \mathbb{R} direction (namely $\tau \mapsto \tau + c$). The action is free if $\alpha \neq 0$ or $\ell \neq 0$.

We denote its quotient space by $\mathring{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$. If $\alpha = 0 = \ell$, we define $\mathring{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ to be the empty set.

We define evaluation maps $\text{ev}_{\pm\infty} : \widehat{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha) \rightarrow M$ by

$$\text{ev}_{\pm\infty}(u) = \lim_{\tau \rightarrow \pm\infty} (\phi_H^t)^{-1}(u(\tau, t)). \quad (29.9)$$

Here we would like to point out that for any $u \in \widehat{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ the right hand side of (29.9) converges to $p_{\pm} \in M$ that is independent of t . Therefore the evaluation map is well-defined. The maps $\text{ev}_{\pm\infty}$ factor through $\mathring{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$.

We define $\text{ev} = (\text{ev}_1, \dots, \text{ev}_{\ell}) : \widehat{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha) \rightarrow M^{\ell}$ as follows.

$$\text{ev}_i(u; z_1^+, \dots, z_{\ell}^+) = \phi_H^{-t}(u(z_i^+)) \quad (29.10)$$

where $z_i^+ = (\tau, t)$. It factors through $\mathring{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ also.

We consider the case $H = 0$ in $\widehat{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ and write it $\widehat{\mathcal{M}}_{\ell}(H = 0, J_0; *, *, \alpha)$. (Note that $J_t^H = J_0$ if $H = 0$.)

Lemma 29.7. $\widehat{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ is isomorphic to $\widehat{\mathcal{M}}_{\ell}(H = 0, J_0; *, *, \alpha)$. The isomorphism is compatible with evaluation maps and \mathbb{R} actions.

Proof. Let $(u; z_1^+, \dots, z_{\ell}^+) \in \mathring{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ we put

$$v(\tau, t) = (\phi_H^t)^{-1}(u(\tau, t)).$$

Then $(v; z_1^{+'}, \dots, z_{\ell}^{+'}) \in \mathring{\mathcal{M}}_{\ell}(H = 0, J_0; *, *, \alpha)$. The assignment $(u; z_1^+, \dots, z_{\ell}^+) \mapsto (v; z_1^{+'}, \dots, z_{\ell}^{+'})$ gives the required isomorphism. \square

We can prove that $\mathring{\mathcal{M}}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ and $\mathring{\mathcal{M}}_{\ell}(H = 0, J_0; *, *, \alpha)$ have compactifications $\mathcal{M}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$ and $\mathcal{M}_{\ell}(H = 0, J_0; *, *, \alpha)$, respectively. They have Kuranishi structures which are isomorphic. We can define an S^1 action on $\mathcal{M}_{\ell}(H = 0, J_0; *, *, \alpha)$ by using the S^1 action on $\mathbb{R} \times S^1$. We then use the isomorphism to define an S^1 action on $\mathcal{M}_{\ell}(H, J^H; z_*^H, z_*^H; \alpha)$. Evaluation maps are compatible with this action. The isotropy group of this S^1 action is always finite. (We note that we have $\alpha \neq 0$ or $\ell \neq 0$ by definition.)

- Lemma 29.8.** (1) *The moduli space $\overset{\circ}{\mathcal{M}}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ has a compactification $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ that is Hausdorff.*
(2) *The space $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ has an orientable Kuranishi structure with corners.*
(3) *The boundary of $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ is described as the union of the following two types of fiber products.*

$$\bigcup \mathcal{M}_{\#\mathbb{L}_1}(H_\chi, J_\chi^H; *, z_*^H; \alpha_1)_{ev_{+\infty}} \times_{ev_{-\infty}} \mathcal{M}_{\#\mathbb{L}_2}(H, J^H; z_*^H, z_*^H; \alpha_2) \quad (29.11)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$ and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$. Here $\alpha_1 + \alpha_2$ is as in (29.4). The fiber product is taken over M .

$$\bigcup \mathcal{M}_{\#\mathbb{L}_1}(H = 0, J_0; *, *, \alpha_1)_{ev_{+\infty}} \times_{ev_{-\infty}} \mathcal{M}_{\#\mathbb{L}_2}(H_\chi, J_\chi^H; *, z_*^H; \alpha_2) \quad (29.12)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$ and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$. The fiber product is taken over M .

- (4) *We may choose $\alpha_0 \in \Pi_2(M; H)$ such that the (virtual) dimension satisfies the following equality (29.13).*

$$\dim \mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha_0 + \alpha) = 2c_1(M) \cap \alpha + 2n + 2\ell. \quad (29.13)$$

- (5) *We can define orientations of $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ so that (3) above are compatible with this orientation.*
(6) *Evaluation maps extend to $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ in a way compatible with (3) above.*
(7) *The map $ev_{+\infty}$ becomes a weakly submersive map in the sense of [FOOO1] Definition A1.13. Here $ev_{+\infty}$ is defined in the same way as in (29.9).*

Here the compatibility with evaluation maps claimed in (6) above is described as follows. Let us consider the boundary in (29.11). Let $i \in \mathbb{L}_2$ be the j -th element of \mathbb{L}_2 . We have

$$ev_i : \mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha) \rightarrow E_{\phi_H}$$

and

$$ev_j : \mathcal{M}_{\#\mathbb{L}_2}(H, J^H; z_*^H, z_*^H; \alpha) \rightarrow M.$$

Denote by t the second coordinate of the marked point in $\mathbb{R} \times S^1$. Then $(\phi_H^t)^{-1} \circ ev_j$ is equal to second factor of the ev_i with respect to $U_3 \cong D_+ \times M$.

The proof of Lemma 29.8 is the same as the proof of Proposition 3.6 and is omitted. (Note the end of $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ where an element of $\overline{\mathcal{M}}(0, J_0; *, *, \alpha)$ (that is the case when $H = 0$ in $\overline{\mathcal{M}}(H_\chi, J_\chi^H; *, z_*^H; \alpha)$) bubbles at $\tau \rightarrow -\infty$ may be regarded as codimension 2 because of S^1 symmetry.

To define operators which include bulk deformations we need the following result due to Lalonde-McDuff-Polterovich [LMP].

Theorem 29.9 (Lalonde-McDuff-Polterovich). *There exists a section*

$$H^*(M; \mathbb{C}) \rightarrow H^*(E_{\phi_H}; \mathbb{C})$$

to the \mathbb{C} linear map $H^(E_{\phi_H}; \mathbb{C}) \rightarrow H^*(M; \mathbb{C})$ induced by the inclusion.*

Remark 29.10. (1) Theorem 29.9 is [LMP] Theorem 3B. We give a proof of Theorem 29.9 in Subsection 29.3 for completeness. The proof we give in Subsection 29.3 is basically the same as the one in [LMP].

- (2) The proof by [LMP] as well as our proof in Subsection 29.3 uses the construction which is closely related to the definition of Seidel homomorphism. We use Theorem 29.9 to define Seidel homomorphism with bulk. However the argument is not circular by the following reason. We do *not* use Theorem 29.9 to define Seidel homomorphism in the case when the bulk deformation \mathfrak{b} is zero. The proof of Theorem 29.9 uses the construction of Seidel homomorphism *without bulk* only, that is the case $\mathfrak{b} = 0$.

Consider a system of continuous families of multisections of $\mathcal{M}_\ell(H, J_H; z_*^H, z_*^H; \alpha)$ and of $\mathcal{M}_\ell(H = 0, J_0; *, *; \alpha)$ which are transversal to 0, S^1 -equivariant and is compatible with the isomorphism in Lemma 29.7. Moreover we may assume that it is compatible with the identification

$$\begin{aligned} & \partial \mathcal{M}_\ell(H, J^H; z_*^H, z_*^H; \alpha) \\ &= \bigcup \mathcal{M}_{\#\mathbb{L}_1}(H, J^H; z_*^H, z_*^H; \alpha_1)_{\text{ev}+\infty} \times_{\text{ev}-\infty} \mathcal{M}_{\#\mathbb{L}_2}(H, J^H; z_*^H, z_*^H; \alpha_2) \end{aligned} \quad (29.14)$$

of the boundary. Furthermore we may assume that the evaluation map $\text{ev}+\infty$ is a submersion on the zero set of this families of multisections.

Then, there exists a system of continuous families of multisections of the moduli space $\mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ such that they are transversal to 0, compatible with the description of the boundary in Lemma 29.8 (3) and that $\text{ev}+\infty$ is a submersion on its zero set.

Let $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_2 + \mathfrak{b}_+$ be as in (5.5). We use Theorem 29.9 to regard them as de Rham cohomology classes of E_{ϕ_H} and denote them as $\widehat{\mathfrak{b}}_2, \widehat{\mathfrak{b}}_+$.

Now we define

$$\mathcal{S}_{(H_\chi, J_\chi)}^{\mathfrak{b}} : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow \Omega(M) \widehat{\otimes} \Lambda^\downarrow$$

as follows. Let $h \in \Omega(M)$. We put

$$\mathcal{S}_{(H_\chi, J_\chi); \alpha}^{\mathfrak{b}}(h) = \sum_{\ell=0}^{\infty} \frac{\exp(\int_\alpha \widehat{\mathfrak{b}}_2)}{\ell!} \text{ev}+\infty! (\text{ev}^*(\underbrace{\widehat{\mathfrak{b}}_+, \dots, \widehat{\mathfrak{b}}_+}_{\ell}) \wedge \text{ev}^*_{-\infty} h)$$

where we use

$$(\text{ev}; \text{ev}-\infty, \text{ev}+\infty) : \mathcal{M}_\ell(H_\chi, J_\chi^H; *, z_*^H; \alpha) \rightarrow E_{\phi_H}^\ell \times M^2.$$

We define $\int_\alpha \widehat{\mathfrak{b}}_2$ as follows. Let $u \in \mathring{\mathcal{M}}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha)$. It induces a map $\widehat{u} : \mathbb{C}P^1 \rightarrow E_{\phi_H}$. We put

$$\int_\alpha \widehat{\mathfrak{b}}_2 = \int_{\mathbb{C}P^1} \widehat{u}^* \widehat{\mathfrak{b}}_2.$$

It is easy to see that it depends only on α and is independent of the representative u .

Let $u \in \mathring{\mathcal{M}}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ and $p = \text{ev}+\infty(u)$. Then $[z_p^H, u] \in \text{Crit}(\mathcal{A}_H)$. We put

$$\mathcal{A}_H(\alpha) = \mathcal{A}_H([z_p^H, u]).$$

We then define

$$\mathcal{S}_{(H_\chi, J_\chi)}^{\mathfrak{b}} = \sum_{\alpha} q^{\mathcal{A}_H(\alpha)} \mathcal{S}_{(H_\chi, J_\chi); \alpha}^{\mathfrak{b}}.$$

Lemma 29.11. $\mathcal{A}_H([z_p^H, u])$ is independent of u but depends only on α .

Proof. Recall that $I_+(\tau, t, x) = (e^{-2\pi(\tau-1+\sqrt{-1}t)}, (\phi_H^t)^{-1}(x))$ is the map $: U_2 \rightarrow U_3$. It is easy to see that

$$I_+^* \Omega = \omega$$

where ω is the pull back of the symplectic form of M to U_2, U_3 and Ω is as in Lemma 29.2.

We have

$$\int \widehat{u}^* \Omega = \int u^* \omega + \int H_t(z_p^H(t)) dt = -\mathcal{A}_H([(z_p^H, w)]).$$

The lemma follows from Stokes' theorem. □

Lemma 29.12.

$$\mathcal{S}_{(H_\chi, J_\chi^H),*}^b \circ d = d \circ \mathcal{S}_{(H_\chi, J_\chi^H),*}^b.$$

The proof is immediate from Lemma 29.8 (3). Thus we obtain

$$\mathcal{S}_{(H_\chi, J_\chi^H),*}^b : H(M; \Lambda^\downarrow) \rightarrow H(M; \Lambda^\downarrow). \quad (29.15)$$

Theorem 29.13. (1) $\mathcal{S}_{(H_\chi, J_\chi^H),*}^b$ is independent of the family of compatible almost complex structures J_χ^H and other choices involved such as multisection.
 (2) $\mathcal{S}_{(H_\chi, J_\chi^H),*}^b$ depends only on the homotopy class of the loop $t \mapsto \phi_H^t$ in the group of Hamiltonian diffeomorphisms.
 (3) We have

$$\mathcal{S}_{(H_\chi, J_\chi^H),*}^b(x \cup^b y) = x \cup^b \mathcal{S}_{(H_\chi, J_\chi^H),*}^b(y).$$

(4) Let H_1, H_2 be two time periodic Hamiltonian such that $\psi_{H_1} = \psi_{H_2} = \text{identity}$. Then we have

$$\mathcal{S}_{((H_1 \# H_2)_\chi, J_\chi^{H_1 \# H_2}),*}^b(x \cup^b y) = \mathcal{S}_{((H_1)_{\chi_1}, J_\chi^{H_1}),*}^b(x) \cup^b \mathcal{S}_{((H_2)_\chi, J_\chi^{H_2}),*}^b(y).$$

We define

$$\mathcal{S}^b : \pi_1(\text{Ham}(M, \omega)) \rightarrow H(M, \Lambda^\downarrow)$$

by

$$\mathcal{S}^b([\phi_H]) = \mathcal{S}_{(H_\chi, J_\chi^H),*}^b(1).$$

Here H is a time dependent Hamiltonian such that $\psi_H = 1$. $[\phi_H]$ is the homotopy class of the loop in $\text{Ham}(M; \omega)$ determined by $t \mapsto \phi_H^t$. 1 is the unit of $H(M; \Lambda^\downarrow)$. (Note that 1 is also the unit with respect to the quantum cup product on $QH_b(M; \Lambda^\downarrow)$ with the bulk.)

The proof of Theorem 29.13 will be given in Subsection 29.2 for completeness.

Corollary 29.14. \mathcal{S}^b is a homomorphism to the group $QH_b(M; \Lambda^\downarrow)^\times$ of invertible elements of $QH_b(M; \Lambda^\downarrow)$.

Definition 29.15. We call the representation

$$\mathcal{S}^b : \pi_1(\text{Ham}(M; \omega)) \rightarrow QH_b(M; \Lambda^\downarrow)^\times.$$

Seidel homomorphism with bulk.

Remark 29.16. As mentioned before the homomorphism \mathcal{S}^b is obtained by Seidel [Se] in the case $b = 0$ under certain hypothesis on the symplectic manifold (M, ω) . Once the virtual fundamental chain technique had been established in the year 1996, it is obvious that we can generalize [Se] to arbitrary (M, ω) . The generalization to include bulk deformations is also straightforward and do not require novel ideas.

Proof. We prove Corollary 29.14 assuming Theorem 29.13. Let $[\phi_{H_i}] \in \pi_1(\text{Ham}(M; \omega))$. We have $[\phi_{H_1 \# H_2}] = [\phi_{H_2}][\phi_{H_1}]$. Then using Theorem 29.13 (3),(4) we have:

$$\begin{aligned} \mathcal{S}^b([\phi_{H_1 \# H_2}]) &= \mathcal{S}^b_{((H_1 \# H_2)_\chi, J_\chi^{H_1 \# H_2}), *}(1) \\ &= \mathcal{S}^b_{((H_1)_\chi, J_\chi^{H_1}), *}(1) \cup^b \mathcal{S}^b_{((H_2)_\chi, J_\chi^{H_2}), *}(1) = \mathcal{S}^b([\phi_{H_1}]) \cup^b \mathcal{S}^b([\phi_{H_2}]). \end{aligned}$$

Thus \mathcal{S}^b is a homomorphism. It implies in particular that the elements of the image are invertible. \square

29.2. Proof of Theorem 29.13. The proof of Theorem 29.13 (1),(2) is similar to the proof of Theorem 7.7 and hence is omitted.

The proof of Theorem 29.13 (3),(4) is similar to the proof of Theorem 11.10 and proceed as follows.

Let Σ be as in Subsection 11.1. We use also the notations $h : \Sigma \rightarrow \mathbb{R}$, $\mathfrak{S} \subset \Sigma$ etc. in Subsection 11.1. We define a Σ parametrized family of almost complex structures J^{H_1, H_2} by $J^{H_1, H_2}(\varphi(\tau, t)) = J_t^{H_1 \# H_2}$. We assume that $(H_1)_t = (H_2)_t = 0$ if t is in a neighborhood of $[0] \in S^1 = \mathbb{R}/\mathbb{Z}$. Let $H^\varphi : \Sigma \times M \rightarrow \mathbb{R}$ be a function as in (11.6).

Definition 29.17. We denote by $\mathring{\mathcal{M}}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \Sigma \rightarrow M$ and $z_i^+ \in \Sigma$, which satisfy the following conditions:

- (1) The map $\bar{u} = u \circ \varphi$ satisfies the equation:

$$\frac{\partial \bar{u}}{\partial \tau} + J^{H_1, H_2} \left(\frac{\partial \bar{u}}{\partial t} - X_{H^\varphi}(\bar{u}) \right) = 0. \quad (29.16)$$

- (2) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial \bar{u}}{\partial \tau} \right|_{J^{H_1, H_2}}^2 + \left| \frac{\partial \bar{u}}{\partial t} - X_{H^\varphi}(\bar{u}) \right|_{J^{H_1, H_2}}^2 \right) dt d\tau$$

is finite.

- (3) There exist $p_{-,1}, p_{-,2}, p_+ \in M$ such that u satisfies the following three asymptotic boundary conditions.

$$\lim_{\tau \rightarrow +\infty} u(\varphi(\tau, t)) = z_{p_+}^{H_1 \# H_2}(t).$$

$$\lim_{\tau \rightarrow -\infty} u(\varphi(\tau, t)) = \begin{cases} z_{p_{-,1}}^{H_1}(2t) & t \leq 1/2, \\ z_{p_{-,2}}^{H_2}(2t-1) & t \geq 1/2. \end{cases}$$

- (4) The homology class of u is α , in the sense we explain below.

- (5) z_1^+, \dots, z_ℓ^+ are mutually distinct.

Here the homology class of u which we mention in (4) above is as follows. We put

$$v(\tau, t) = \begin{cases} (\phi_{2H_1}^t)^{-1}(u(\tau, t)) & \tau \leq 0, 0 \leq t \leq 1/2 \\ (\phi_{2H_2}^{t-1/2})^{-1}(u(\tau, t)) & \tau \leq 0, 1/2 \leq t \leq 1 \\ (\phi_{H_1 \# H_2}^t)^{-1}(u(\tau, t)) & \tau \geq 0. \end{cases} \quad (29.17)$$

It defines a map $\Sigma \rightarrow M$ which extends to a continuous map $v : S^2 \rightarrow M$. (Note that Σ is $S^2 \setminus \{3 \text{ points}\}$.) The homology class of u is by definition $v_*([S^2]) \in H_2(M; \mathbb{Z})$.

We denote by

$$(\text{ev}_{-\infty,1}, \text{ev}_{-\infty,2}, \text{ev}_{+\infty}) : \mathring{\mathcal{M}}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha) \rightarrow M^3$$

the map which associates $(p_{-,1}, p_{-,2}, p_+)$ to $(u; z_1^+, \dots, z_\ell^+)$. We also define an evaluation map

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha) \rightarrow M^\ell$$

that associates to $(u; z_1^+, \dots, z_\ell^+)$ the point $(u(z_1^+), \dots, u(z_\ell^+))$.

- Lemma 29.18.** (1) *The moduli space $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ has a compactification $\mathring{\mathcal{M}}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ that is Hausdorff.*
(2) *The space $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ has an orientable Kuranishi structure with corners.*
(3) *The boundary of $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ is described by the union of the following three types of fiber products.*

$$\mathcal{M}_{\# \mathbb{L}_1}(H_1, J^{H_1}; z_*^{H_1}, z_*^{H_1}; \alpha_1) \times_{\text{ev}_{+\infty}} \times_{\text{ev}_{-\infty, 1}} \mathcal{M}_{\# \mathbb{L}_2}(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha_2) \quad (29.18)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\mathcal{M}_{\# \mathbb{L}_1}(H_2, J^{H_2}; z_*^{H_2}, z_*^{H_2}; \alpha_1) \times_{\text{ev}_{+\infty}} \times_{\text{ev}_{-\infty, 2}} \mathcal{M}_{\# \mathbb{L}_2}(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha_2) \quad (29.19)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\begin{aligned} & \mathcal{M}_{\# \mathbb{L}_2}(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha_1) \\ & \times_{\text{ev}_{+\infty}} \times_{\text{ev}_{-\infty}} \mathcal{M}_{\# \mathbb{L}_1}(H_1 \# H_2, J^{H_1 \# H_2}; z_*^{H_1 \# H_2}, z_*^{H_1 \# H_2}; \alpha_2) \end{aligned} \quad (29.20)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

- (4) *The (virtual) dimension satisfies the following equality:*

$$\dim \mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha) = 2\ell + 2c_1(M)[\alpha] + 2n. \quad (29.21)$$

- (5) *We can define orientations of $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ so that (3) above is compatible with this orientation.*
(6) *$\text{ev}_{-\infty, 1}, \text{ev}_{-\infty, 2}, \text{ev}_{+\infty}, \text{ev}$ extend to strongly continuous smooth maps on $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$, which we denote also by the same symbol. They are compatible with (3).*
(7) *$\text{ev}_{+\infty}$ is weakly submersive.*

We take a system of families of multisections on $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ that are transversal to 0, compatible with (3) above and such that the restriction of $\text{ev}_{+\infty}$ to its zero set is a submersion.

We define

$$\mathbf{m}_{2; \alpha}^{\text{cl}, H^\varphi} : \Omega(M) \otimes \Omega(M) \rightarrow \Omega(M)$$

by

$$\begin{aligned} & \mathbf{m}_{2; \alpha}^{\text{cl}; \mathbf{b}; H^\varphi}(h_1, h_2) \\ &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \text{ev}_{+\infty}^* \left(\text{ev}_{-\infty, 1}^* h_1 \wedge \text{ev}_{-\infty, 2}^* h_2 \wedge \text{ev}^* \underbrace{(\mathbf{b}_+, \dots, \mathbf{b}_+)}_{\ell} \right), \end{aligned} \quad (29.22)$$

where we use $\mathcal{M}_\ell(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha)$ and evaluation maps on it to define the right hand side.

We put

$$\mathfrak{m}_2^{\text{cl}, \mathfrak{b}; H^\varphi} = \sum_{\alpha} e^{\alpha \cap \widehat{\mathfrak{b}}_2} q^{-\alpha \cap \omega} \mathfrak{m}_{2; \alpha}^{\text{cl}, \mathfrak{b}; H^\varphi}.$$

It is a chain map, since the contribution of the boundaries described in (3) above to the correspondence are all zero by the S^1 equivariance. Therefore $\mathfrak{m}_2^{\text{cl}, \mathfrak{b}; H^\varphi}$ defines a map

$$\mathfrak{m}_2^{\text{cl}, \mathfrak{b}; H^\varphi} : H(M; \Lambda^\downarrow) \otimes_{\Lambda^\downarrow} H(M; \Lambda^\downarrow) \rightarrow H(M; \Lambda^\downarrow).$$

Lemma 29.19.

$$\mathfrak{m}_2^{\text{cl}, \mathfrak{b}; H^\varphi} \circ \left(\mathcal{S}_{((H_1)_\chi, J_\chi^{H_1})}^{\mathfrak{b}} \otimes \mathcal{S}_{((H_2)_\chi, J_\chi^{H_2})}^{\mathfrak{b}} \right)$$

is chain homotopic to

$$\mathcal{S}_{((H_1 \# H_2)_\chi, J_\chi^{H_1 \# H_2})}^{\mathfrak{b}} \circ \cup^{\mathfrak{b}}.$$

Proof. For $S \in \mathbb{R}$ we define $H_{S, \chi}^\varphi : \Sigma \times M \rightarrow \mathbb{R}$ by

$$H_{S, \chi}^\varphi(\varphi(\tau, t), x) = \chi(\tau + S)(H_1 \# H_2)_t(x). \quad (29.23)$$

Note that J_χ^H is the $\mathbb{R} \times S^1$ parametrized family of almost complex structures as in (29.6). For $S \in \mathbb{R}$ we define a Σ parametrized family of almost complex structures $J_{S, \chi}^{H_1, H_2}$ by

$$J_{S, \chi}^{H_1, H_2}(\varphi(\tau, t)) = J_\chi^{H_1 \# H_2}(\tau + S, t). \quad (29.24)$$

Let $\alpha \in \Pi_2(M; H_1 \# H_2)$.

Definition 29.20. For $S \in \mathbb{R}$, we denote by $\mathring{\mathcal{M}}_\ell(H_{S, \chi}^\varphi, J_{S, \chi}^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha)$ the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \Sigma \rightarrow M$ and $z_i^+ \in \Sigma$, which satisfy the following conditions:

- (1) The map $\overline{u} = u \circ \varphi$ satisfies the equation:

$$\frac{\partial \overline{u}}{\partial \tau} + J_{S, \chi}^{H_1, H_2} \left(\frac{\partial \overline{u}}{\partial t} - X_{H_{S, \chi}^\varphi}(\overline{u}) \right) = 0. \quad (29.25)$$

- (2) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial \overline{u}}{\partial \tau} \right|_{J_{S, \chi}^{H_1, H_2}}^2 + \left| \frac{\partial \overline{u}}{\partial t} - X_{H_{S, \chi}^\varphi}(\overline{u}) \right|_{J_{S, \chi}^{H_1, H_2}}^2 \right) dt d\tau$$

is finite.

- (3) There exist $p_{-,1}, p_{-,2}, p_+ \in M$ such that u satisfies the following three asymptotic boundary conditions.

$$\lim_{\tau \rightarrow +\infty} u(\varphi(\tau, t)) = z_{p_+}^{H_1 \# H_2}(t).$$

$$\lim_{\tau \rightarrow -\infty} u(\varphi(\tau, t)) = \begin{cases} p_{-,1} & t \leq 1/2, \\ p_{-,2} & t \geq 1/2. \end{cases}$$

- (4) The homology class of u is α , in the sense we explain below.

- (5) z_1^+, \dots, z_ℓ^+ are mutually distinct.

Here the homology class of u which we mention in (4) above is as follows. We consider $\Sigma \times M$ and glue M at the two ends corresponding to $\tau \rightarrow -\infty$ by $(\phi_{H_1}^{2t})^{-1}$ $0 \leq t \leq 1/2$ and $(\phi_{H_2}^{2t-1})^{-1}$, $1/2 \leq t \leq 1$, respectively. At the end corresponding to $\tau \rightarrow +\infty$ we glue M but with twisting using the map $\phi_{H_1 \# H_2}$ in the same way

as the definition of E_{H^φ} . We then obtain E_{H^φ} . Actually this space together with projection to $S^2 = \Sigma \cup \{3 \text{ points}\}$ can be identified with $E_{\phi_{H_1 \# H_2}}$. We define

$$\widehat{u}(\tau, t) = ((\tau, t), u(\tau, t)) \in E_{H^\varphi}. \quad (29.26)$$

It extends to a continuous map $\widehat{u} : S^2 \rightarrow E_{H^\varphi}$. The homology class of \widehat{u} is well defined as an element of $\Pi_2(M; H_1 \# H_2)$.

By Theorem 29.9 we obtain $\widehat{\mathbf{b}}_2 \in H^2(E_{H^\varphi}; \mathbb{C})$ from $\mathbf{b}_2 \in H^2(M; \mathbb{C})$.

We denote by

$$(\text{ev}_{-\infty,1}, \text{ev}_{-\infty,2}, \text{ev}_{+\infty}) : \mathring{\mathcal{M}}_\ell(H_{S,\chi}^\varphi, J_{S,\chi}^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha) \rightarrow M^3$$

the map which associate $(p_{-,1}, p_{-,2}, p_+)$ to $(u; z_1^+, \dots, z_\ell^+)$. We also define an evaluation map

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_\ell) : \mathring{\mathcal{M}}_\ell(H_{S,\chi}^\varphi, J_{S,\chi}^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha) \rightarrow (E_{H^\varphi})^\ell$$

that associates to $(u; z_1^+, \dots, z_\ell^+)$ the point $(\widehat{u}(z_1^+), \dots, \widehat{u}(z_\ell^+))$. We put

$$\mathring{\mathcal{M}}_\ell(\text{para}; H_\chi^\varphi, J_\chi^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha) = \bigcup_{S \in \mathbb{R}} \{S\} \times \mathring{\mathcal{M}}_\ell(H_{S,\chi}^\varphi, J_{S,\chi}^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha).$$

The above evaluation maps are defined on it in an obvious way.

We can define a compactification $\mathcal{M}_\ell(\text{para}; H_\chi^\varphi, J_\chi^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha)$ of the moduli space $\mathring{\mathcal{M}}_\ell(\text{para}; H_\chi^\varphi, J_\chi^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha)$ and a system of Kuranishi structures on it, that are oriented with corners. Its boundary is a union of the following five types of fiber products.

$$\begin{aligned} & \mathcal{M}_{\# \mathbb{L}_1}(H = 0, J_0; *, *, \alpha_1) \\ & \text{ev}_{+\infty} \times_{\text{ev}_{-\infty,1}} \mathcal{M}_{\# \mathbb{L}_2}(\text{para}; H_\chi^\varphi, J_\chi^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha_2), \end{aligned} \quad (29.27)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\begin{aligned} & \mathcal{M}_{\# \mathbb{L}_1}(H = 0, J_0; *, *, \alpha_1) \\ & \text{ev}_{+\infty} \times_{\text{ev}_{-\infty,2}} \mathcal{M}_{\# \mathbb{L}_2}(\text{para}; H_\chi^\varphi, J_\chi^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha_2), \end{aligned} \quad (29.28)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\begin{aligned} & \mathcal{M}_{\# \mathbb{L}_1}(\text{para}; H_\chi^\varphi, J_\chi^{H_1, H_2}; **, z_*^{H_1 \# H_2}; \alpha_1) \\ & \text{ev}_{+\infty} \times_{\text{ev}_{-\infty}} \mathcal{M}_{\# \mathbb{L}_2}(H_1 \# H_2, J^{H_1 \# H_2}, z_*^{H_1 \# H_2}, z_*^{H_1 \# H_2}; \alpha_2), \end{aligned} \quad (29.29)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\mathcal{M}_{\# \mathbb{L}_1+3}^{\text{cl}}(\alpha_1)_{\text{ev}_3} \times_{\text{ev}_{-\infty}} \mathcal{M}_{\# \mathbb{L}_2}((H_1 \# H_2)_\chi, J_\chi^{H_1 \# H_2}; *, z_*^{H_1 \# H_2}; \alpha_2), \quad (29.30)$$

where the union is taken over all α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$, and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$.

$$\begin{aligned} & \mathcal{M}_{\# \mathbb{L}_1}((H_1)_\chi; J_\chi^{H_1}, z_*^{H_1}; \alpha_1) \times \mathcal{M}_{\# \mathbb{L}_2}((H_2)_\chi; J_\chi^{H_2}; *, z_*^{H_2}; \alpha_2) \\ & (\text{ev}_{+\infty}, \text{ev}_{+\infty}) \times (\text{ev}_{-\infty,1}, \text{ev}_{-\infty,2}) \mathcal{M}_{\# \mathbb{L}_3}(H^\varphi, J^{H_1, H_2}; z_*^{H_1}, z_*^{H_2}, z_*^{H_1 \# H_2}; \alpha_3), \end{aligned} \quad (29.31)$$

where the union is taken over all $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 = \alpha$ and ‘triple shuffle’ $(\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3)$ of $\{1, \dots, \ell\}$.

Note that (29.27), (29.28), (29.29) are the ends which appear while S is bounded. (29.30) and (29.31) correspond to the case $S \rightarrow -\infty$ and $S \rightarrow +\infty$, respectively.

We next take a system of continuous families of multisections on the moduli space $\mathcal{M}_\ell(\text{para}; H_\chi^\varphi, J_\chi^{H_1 \# H_2}; **, z_*^{H_1 \# H_2}; \alpha)$. We take our continuous families of

multisections so that it is transversal to 0 and the evaluation map $\text{ev}_{+\infty}$ is a submersion on its zero set. Moreover we assume that it is compatible with the above description of the boundary. We remark that the first factor of (29.27), (29.28) and the second factor of (29.29) have S^1 actions so that the isotropy group is finite. So we may take our families of multisections so that it is S^1 equivariant on those factors. Then the contribution of (29.27), (29.28), (29.29) becomes zero when we consider the correspondence by our moduli space.

We use $\mathcal{M}_\ell(\text{para}; H_\chi^\varphi, J_\chi^{H_1 \# H_2}; **, z_*^{H_1 \# H_2}; \alpha)$ and evaluation maps in a way similar to (29.22) to obtain an operator

$$\mathfrak{H} : (\Omega(M) \hat{\otimes} \Lambda^\downarrow) \otimes (\Omega(M) \hat{\otimes} \Lambda^\downarrow) \rightarrow \Omega(M) \hat{\otimes} \Lambda^\downarrow.$$

We have

$$\begin{aligned} & d \circ \mathfrak{H} + \mathfrak{H} \circ d \\ &= \mathfrak{m}_{2;\alpha}^{\text{cl}; \mathfrak{b}; H^\varphi} \circ \left(\mathcal{S}_{((H_1)_\chi, J_\chi^{H_1})}^{\mathfrak{b}} \otimes \mathcal{S}_{((H_2)_\chi, J_\chi^{H_2})}^{\mathfrak{b}} \right) - \mathcal{S}_{((H_1 \# H_2)_\chi, J_\chi^{H_1 \# H_2})}^{\mathfrak{b}} \circ \cup^{\mathfrak{b}}. \end{aligned} \quad (29.32)$$

In fact, (29.30) and (29.31) correspond to the first and second term of the right hand side. The proof of Lemma 29.19 is complete. \square

Theorem 29.13 (4) follows from Lemma 29.19.

To prove (3) we apply (4) to the case $H_1 = 0$, $H_2 = H$. Then using the fact that $\mathcal{S}_{((H_1)_\chi, J_\chi^{H_1}),*}^{\mathfrak{b}} = \text{id}$ we have

$$\mathcal{S}_{((0 \# H)_\chi, J_\chi^{0 \# H}),*}(x \cup^{\mathfrak{b}} y) = x \cup^{\mathfrak{b}} \mathcal{S}_{(H_\chi, J_\chi^H),*}(y) \quad (29.33)$$

in cohomology. We note that $0 \# H$ is the same as H up to change of the coordinate of S^1 . Therefore $\mathcal{S}_{((0 \# H)_\chi, J_\chi^{0 \# H}),*} = \mathcal{S}_{(H_\chi, J_\chi^H),*}$ can be proved by using homotopy between them. (29.33) now implies Theorem 29.13 (3).

Therefore the proof of Theorem 29.13 is now complete. \square

29.3. Proof of Theorem 29.9. As we mentioned before, we did not use Theorem 29.9 in the definition of $\mathcal{S}_{(H_\chi, J_\chi^H)}^0$ or the proof of Theorem 29.13, in the case $\mathfrak{b} = 0$. We will use that case in this subsection.

We consider the moduli space $\mathcal{M}_1(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ to define

$$\mathcal{R}_{(H_\chi, J_\chi^H)} : \Omega(M) \hat{\otimes} \Lambda^\downarrow \rightarrow \Omega(E_{\phi_H}) \hat{\otimes} \Lambda^\downarrow$$

as follows. Let $h \in \Omega(M)$. We put

$$\mathcal{R}_{(H_\chi, J_\chi^H), \alpha}(h) = \text{ev}_{1,!}(\text{ev}_{-\infty}^*(h))$$

where we use the evaluation maps $\text{ev}_1 : \mathcal{M}_1(H_\chi, J_\chi^H; *, z_*^H; \alpha) \rightarrow E_{\phi_H}$ and $\text{ev}_{-\infty} : \mathcal{M}_1(H_\chi, J_\chi^H; *, z_*^H; \alpha) \rightarrow M$. We then put

$$\mathcal{R}_{(H_\chi, J_\chi^H)} = \sum_{\alpha} q^{A_H(\alpha)} \mathcal{R}_{(H_\chi, J_\chi^H), \alpha}.$$

We consider the inclusion map $i_\infty : M \rightarrow E_{\phi_H}$ to the fiber of $\infty \in \mathbb{C}P^1$. By definition it is easy to see that $i_\infty^* \circ \mathcal{R}_{(H_\chi, J_\chi^H)}$ is chain homotopic to $\mathcal{S}_{(H_\chi, J_\chi^H)}$, where $(\phi_H)_*(\gamma(t)) = (\phi_H^t)^{-1}(\gamma(t))$. Note that, if $\gamma(t)$ is a one-periodic orbit of ϕ_H^t , $(\phi_H)^*(\gamma(t))$ is a constant. Therefore

$$i_\infty^* \circ \mathcal{R}_{(H_\chi, J_\chi^H),*} = \mathcal{S}_{(H_\chi, J_\chi^H),*} \quad (29.34)$$

in homology. It follows that for $a \in H(M; \Lambda^\perp)$ we have

$$\begin{aligned} i_\infty^*(\mathcal{R}_{(H_\chi, J_\chi^H),*}(a \cup^Q \mathcal{S}(\tilde{\psi}_H)^{-1})) &= \mathcal{S}_{(H_\chi, J_\chi^H),*}(a^b) \cup^Q \mathcal{S}(\tilde{\psi}_H)^{-1} \\ &= a \cup^Q \mathcal{S}(\tilde{\psi}_H) \cup^Q \mathcal{S}(\tilde{\psi}_H)^{-1} = a. \end{aligned} \quad (29.35)$$

Thus

$$a \mapsto \hat{a} = \mathcal{R}_{(H_\chi, J_\chi^H),*}(a \cup^Q \mathcal{S}(\tilde{\psi}_H)^{-1})$$

is a required section. The proof of Theorem 29.9 is complete. \square

30. SPECTRAL INVARIANTS AND SEIDEL HOMOMORPHISM

In this section we study the relationship between Seidel homomorphism and spectral invariants.

30.1. Valuations and spectral invariants. The next theorem is a straightforward generalization of the result Theorem 4.3 [Oh2] and Proposition 4.1 [EP1].

Let H be a time-dependent normalized Hamiltonian such that $\psi_H = id$ and, let $\tilde{\psi}_H$ be an associated element of $\pi_1(\text{Ham}(M; \omega))$.

Theorem 30.1. *For each $a \in QH_b(M; \Lambda^\perp)$ we have*

$$\rho^b(H; a) = \mathfrak{v}_q(a \cup^b \mathcal{S}^b(\tilde{\psi}_H)).$$

Proof. The proof is similar to the proof of Theorem 9.1. Let H_k be a sequence of normalized time dependent Hamiltonians such that ψ_{H_k} are nondegenerate and $\lim_{k \rightarrow \infty} H_k = H$ in C^0 topology. We put

$$F_k^\chi(\tau, t, x) = H_k(t, x) + \chi(\tau)(H(t, x) - H_k(t, x)) : \mathbb{R} \times S^1 \times M \rightarrow \mathbb{R}. \quad (30.1)$$

We fix J_0 and define $J^{H_k}, J^H, J_\chi^{H_k}, J_\chi^H$ as in (29.5) and (29.6). Let J_k^χ be an $\mathbb{R} \times S^1$ parametrized family of almost complex structures such that

$$J_k^\chi(\tau, t) = \begin{cases} J_t^H & \tau \geq 2, \\ J_t^{H_k} & \tau \leq -2. \end{cases}$$

Let $[\gamma, w] \in \text{Crit}(\mathcal{A}_{H_k})$ and $\alpha \in \Pi_2(M; H)$.

Definition 30.2. We denote by $\mathring{\mathcal{M}}_\ell(F_k^\chi, J_{F_k}^\chi; [\gamma, w], *, \alpha)$ the set of all pairs $(u; z_1^+, \dots, z_\ell^+)$ of maps $u : \mathbb{R} \times S^1 \rightarrow M$ and $z_i^+ \in \mathbb{R} \times S^1$ which satisfy the following conditions:

(1) The map u satisfies the equation:

$$\frac{\partial u}{\partial \tau} + J_k^\chi \left(\frac{\partial u}{\partial t} - X_{F_k^\chi}(u) \right) = 0. \quad (30.2)$$

(2) The energy

$$\frac{1}{2} \int \left(\left| \frac{\partial u}{\partial \tau} \right|_{J_k^\chi}^2 + \left| \frac{\partial u}{\partial t} - X_{F_k^\chi}(u) \right|_{J_k^\chi}^2 \right) dt d\tau$$

is finite.

(3) There exists p such that the following asymptotic boundary condition is satisfied.

$$\lim_{\tau \rightarrow -\infty} u(\tau, t) = \gamma(t), \quad \lim_{\tau \rightarrow +\infty} u(\tau, t) = z_p^H(t).$$

(4) The homology class of $w \# u$ is α , where $\#$ is the obvious concatenation.

(5) z_i^+ are distinct to each other.

The space $\overset{\circ}{\mathcal{M}}_\ell(F_k^\chi, J_k^\chi; [\gamma, w], *, \alpha)$ has a compactification $\mathcal{M}_\ell(F_k^\chi, J_k^\chi; [\gamma, w], *, \alpha)$, on which there exists a system of oriented Kuranishi structure with corners which is compatible at the boundaries. There exists a system of multisections of this Kuranishi structures so that the map $(u; z_1^+, \dots, z_\ell^+) \mapsto \lim_{\tau \rightarrow +\infty} u(\tau, 0)$ defines an weakly submersive map $\mathcal{M}_\ell(F_k^\chi, J_k^\chi; [\gamma, w], *, \alpha) \rightarrow M$.

We use it in a way similar to the argument we did several times to define a map

$$\mathcal{P}_{(F_k^\chi, J_k^\chi)}^b : CF(M, H_k, J; \Lambda^\downarrow) \rightarrow \Omega(M) \widehat{\otimes} \Lambda^\downarrow.$$

Here we identify $\Omega(M) \otimes \Lambda^\downarrow$ and the Floer chain module $CF(M, H, J; \Lambda^\downarrow)$ of Bott-Morse type using the Hamiltonian loop $\{\phi_H^t\}$.

Lemma 30.3. $\mathcal{P}_{(F_k^\chi, J_k^\chi)}^b \circ \mathcal{P}_{((H_k)_\chi, J_\chi^{H_k})}^b$ is chain homotopic to $\mathcal{S}_{(H_\chi, J_\chi^H)}^b$

The proof is similar to the proof of Lemma 9.6 and is omitted.

Lemma 30.4.

$$\mathcal{P}_{(F_k^\chi, J_k^\chi)}^b (F^\lambda CF(M, H_k, J; \Lambda^\downarrow)) \subset \Omega(M) \widehat{\otimes} q^{\lambda+E^-(H-H_k)} \Lambda_0^\downarrow.$$

The proof is similar to the proof of Lemma 9.8 and (9.18) and so it omitted. Lemmas 30.4 implies

$$\mathfrak{v}_q(\mathcal{S}_{(H_\chi, J_\chi)}^b(a)) \leq \rho^b(H_k; a) + E^-(H - H_k).$$

Taking the limit $k \rightarrow \infty$ we have

$$\rho^b(H; a) \geq \mathfrak{v}_q(\mathcal{S}_{(H_\chi, J_\chi)}^b(a)). \quad (30.3)$$

We can prove the opposite inequality by using

$$\mathcal{Q}_{(F_k^\chi, J_{F_k}^\chi)}^b : \Omega(M) \widehat{\otimes} \Lambda^\downarrow \rightarrow CF(M, H_k, J; \Lambda^\downarrow)$$

that can be defined in a similar way as Definition 26.7. (See the proof of Proposition 26.10 also.)

By Theorem 29.13 (3) with $x = a, y = 1$, we find that

$$\mathcal{S}_{(H_\chi, J_\chi^H), *}^b(a) = a \cup^b \mathcal{S}^b(\tilde{\psi}_H).$$

The proof of Theorem 30.1 is now complete. \square

Let H be a time dependent periodic Hamiltonian such that $\psi_H = id$. We do not assume that H is normalized. Let $e \in QH_b(M; \omega)$ with $e \cup^b e = e$. We assume that $e\Lambda^\downarrow$ is a direct product factor of $QH_b(M; \omega)$.

Corollary 30.5. We put $e \cup^b \mathcal{S}^b(\tilde{\psi}_H) = xe$ with $x \in \Lambda^\downarrow$. Then

$$\zeta^b(H; e) = -\mathfrak{v}_q(x) + \frac{1}{\text{vol}_\omega(M)} \int_{[0,1]} \int_M H_t dt \omega^n.$$

Proof. We put

$$\underline{H}_t = H_t - \frac{1}{\text{vol}_\omega(M)} \int_M H_t \omega^n.$$

It is a normalized Hamiltonian and $\psi_{\underline{H}} = \psi_H$. By Theorem 30.1 we have

$$\rho^b(\underline{H}; e) = \mathfrak{v}_q(e \cup^b \mathcal{S}^b(\tilde{\psi}_H)).$$

On the other hand,

$$\rho^{\mathfrak{b}}(H; e) = \rho^{\mathfrak{b}}(\underline{H}; e) - \frac{1}{\text{vol}_{\omega}(M)} \int_{[0,1]} \int_M H_t dt \omega^n.$$

Therefore we have

$$\begin{aligned} \zeta^{\mathfrak{b}}(H; e) &= - \left(\lim_{k \rightarrow \infty} \frac{\rho^{\mathfrak{b}}(kH; e)}{k} \right) \\ &= - \lim_{k \rightarrow \infty} \frac{\rho^{\mathfrak{b}}(k\underline{H}; e)}{k} + \frac{1}{\text{vol}_{\omega}(M)} \int_{[0,1]} \int_M H_t dt \omega^n \\ &= - \lim_{k \rightarrow \infty} \frac{\mathfrak{v}_q(ex^k)}{k} + \frac{1}{\text{vol}_{\omega}(M)} \int_{[0,1]} \int_M H_t dt \omega^n \\ &= -\mathfrak{v}_q(x) + \frac{1}{\text{vol}_{\omega}(M)} \int_{[0,1]} \int_M H_t dt \omega^n, \end{aligned}$$

as required. □

30.2. The toric case. In this section we generalize a result by McDuff-Tolman [MT] to a version with bulk and apply the result for some calculation. Our discussion here is a straightforward generalization of [MT].

Let H be a time *independent* normalized Hamiltonian. We assume that $\psi_H = id$. We put

$$H_{\min} = \inf\{H(y) \mid y \in M\}$$

and

$$D_{\min} = \{x \in M \mid H(x) = H_{\min}\}.$$

Since D_{\min} is a connected component of the fixed point set of the S^1 action generated by X_H , it follows that D_{\min} is a smooth submanifold. We assume that D_{\min} is of (real) codimension 2. We also assume the following:

Assumption 30.6. Let $p \in D_{\min}$ and $q \in M \setminus D_{\min}$ be sufficiently close to p . We consider the orbit $z_q^H(t) = \phi_H^t(q)$ and a disk $w : (D^2, \partial D^2) \rightarrow (M, z_q^H)$ which bounds z_q^H . If w is sufficiently small, then

$$[D_{\min}] \cdot w_*(D^2) = +1.$$

Let $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_2 + \mathfrak{b}_+$ as before.

Theorem 30.7. *We have*

$$\mathcal{S}^{\mathfrak{b}}([\phi_H]) \equiv q^{-H_{\min}} e^{\overline{\mathfrak{b}}_2 \cap D_{\min}} PD([D_{\min}]) \mod q^{-H_{\min}} \Lambda_{-}^{\downarrow}.$$

Remark 30.8. In the case $\mathfrak{b} = 0$ this is Theorem 1.9 [MT]. Our generalization to the case $\mathfrak{b} \neq 0$ is actually straightforward.

Proof. We start with the following lemma.

Lemma 30.9. *If $\mathring{\mathcal{M}}_0(H_{\chi}, J_{\chi}^H; *, z_*^H; \alpha)$ is nonempty, then*

$$\mathcal{A}_H(\alpha) \leq -H_{\min}. \tag{30.4}$$

Proof. Let $u \in \mathring{\mathcal{M}}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha)$. Then we have

$$\begin{aligned} \int u^* \omega &= \int \omega \left(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t} \right) dt d\tau = \int \omega \left(\frac{\partial u}{\partial \tau}, J_\chi^H \frac{\partial u}{\partial \tau} + \chi(\tau) X_H \right) dt d\tau \\ &\geq - \int \chi(\tau) \frac{\partial(H \circ u)}{\partial \tau} dt d\tau \\ &\geq - \int_{S^1} H(z_p^H(t)) dt + \int \chi'(\tau) (H \circ u) dt d\tau \\ &\geq - \int_{S^1} H(z_p^H(t)) dt + H_{\min}. \end{aligned}$$

Lemma 30.9 follows. \square

We remark that the equality holds only when

$$\int \omega \left(\frac{\partial u}{\partial \tau}, J_\chi^H \frac{\partial u}{\partial \tau} \right) dt d\tau = \int \left| \frac{\partial u}{\partial \tau} \right|_{J_\chi^H}^2 dt d\tau = 0$$

and so

$$\frac{\partial u}{\partial \tau} = 0.$$

Therefore u must be constant. Moreover since $u(\tau, t) \rightarrow z_p^H(t)$ as $\tau \rightarrow \infty$, the image of u must lie in the zero locus of X_H . Thus

Lemma 30.10. *If the equality holds in Lemma 30.9, $\mathcal{M}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha)$ consists of constant maps to D_{\min} .*

Let α_0 be the homology class such that Lemma 30.9 holds.

Lemma 30.11. *The moduli space $\mathcal{M}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha_0)$ is transversal and*

$$\text{ev}_{1\#}(\mathcal{M}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha_0)) = [D_{\min}].$$

Proof. We consider $D_{\min} \times S^2 \subset E_{\phi_H}$. Its tubular neighborhood is identified with a neighborhood of zero section in the line bundle $D_{\min} \times \mathcal{O}(-1) \rightarrow D_{\min} \times S^2$. Here we identify $S^2 \cong \mathbb{CP}^1$ and $\mathcal{O}(-1)$ is a line bundle with Chern number -1 . (We use Assumption 30.6 here.) The moduli space $\mathcal{M}_0(H_\chi, J_\chi^H; *, z_*^H; \alpha_0)$ then is identified to the moduli space of the sections to the bundle. $D_{\min} \times \mathcal{O}(-1) \rightarrow S^2$. The lemma follows easily. \square

Theorem 30.7 now follows from Lemmas 30.9, 30.10, 30.11. \square

We now specialize Theorem 30.7 to the case of toric manifold. Let (M, ω) be a compact Kähler toric manifold. Then T^n acts on (M, ω) preserving the Kähler form. Let $\pi : M \rightarrow P \subset \mathbb{R}^n$ be the moment map. Let $D_j = \pi^{-1}(\partial_j P)$, $j = 1, \dots, m$ be the irreducible components of the toric divisor. As in Section 20 we have affine functions $\ell_j : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\partial_j P = \{\mathbf{u} \in P \mid \ell_j(\mathbf{u}) = 0\}.$$

We put

$$d\ell_j = (k_{j,1}, \dots, k_{j,n})$$

where $k_{j,1}, \dots, k_{j,n}$ are integers which are coprime. Let S_j^1 be a subgroup of T^n such that

$$S_j^1 = \{[k_{j,1}t, \dots, k_{j,n}t] \mid t \in \mathbb{R}\} \subset T^n$$

where we identify $T^n = \mathbb{R}^n/\mathbb{Z}^n$. We note that if we put $H = \ell_j$ then $\psi_H = \text{identity}$. The next result is a corollary to Theorem 30.7. S_j^1 determines an element of $\pi_1(\text{Ham}(M, \omega))$, which we denote by $[S_j^1]$.

Theorem 30.12.

$$\mathcal{S}^b([S_j^1]) \equiv q^{-\text{vol}(P)^{-1} \int_P \ell_j dq} e^{\bar{\mathbf{b}}_2 \cap D_j} PD([D_j]) \mod q^{-\text{vol}(P)^{-1} \int_P \ell_j dq} \Lambda_-^\perp.$$

Proof. We note that $\ell_j - \text{Vol}(P)^{-1} \int_P \ell_j$ is the normalized Hamiltonian which generates $[S_j^1]$. (This is because the push out measure $\pi_!(\omega^n)$ on P is the Lebeague measure.) Its minimum is attained at D_j . Therefore Theorem 30.12 follows from Theorem 30.7. \square

Let $\mathbf{u}_{\text{cnt}} \in P$ be the center of gravity and $e \in QH_b(M; \Lambda^\perp)$ the idempotent, which corresponds to $\mathbf{u} \in P$ by Theorems 20.17, 20.18 and Proposition 20.22

Theorem 30.13.

$$\mu_e^b([S_j^1]) = \text{Vol}(P)(\ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}})).$$

In particular, $\mu_e^b = 0$ on the image of $\pi_1(T^n) \rightarrow \pi_1(\text{Ham}(M, \omega))$ if and only if $\mathbf{u} = \mathbf{u}_{\text{cnt}}$.

Proof. Let \mathfrak{y} be the critical point of $\mathfrak{P}\mathfrak{D}_b$ that corresponds to $L(\mathbf{u})$. (Namely $\mathbf{u}(\mathfrak{y}) = \mathbf{u}$.) By (20.35), Theorem 20.23, (20.39) and Theorem 21.3 we have

$$i_{\text{qm}, (b, b(\mathfrak{y}))}^*(PD([D_j])) = \left[\frac{\partial \mathfrak{P}\mathfrak{D}_b}{\partial w_j} \right] \equiv e^{\bar{\mathbf{b}}_j} z_j \mod \Lambda_+.$$

Here z_j is as in (21.1). By Lemma 21.2

$$i_{\text{qm}, (b, b(\mathfrak{y}))}^*(e_{\mathfrak{y}}) = 1.$$

We put $b(\mathfrak{y}) = \sum x_i e_i$ and $\partial \beta_j = \sum k_{ji} e_i$, where e_i is a basis of $H(L(\mathbf{u}), \mathbb{Z})$. Then we get

$$z_j(\mathfrak{y}) = q^{\ell_j(\mathbf{u})} \prod_{i=1}^n \exp k_{ji} x_i.$$

Therefore we have

$$i_{\text{qm}, (b, b(\mathfrak{y}))}^*(\mathcal{S}^b([S_j^1])) \equiv q^{\ell_j(\mathbf{u}) - \text{Vol}(P)^{-1} \int_P \ell_j dq} c \mod q^{\ell_j(\mathbf{u}) - \text{Vol}(P)^{-1} \int_P \ell_j dq} \Lambda_-^\perp$$

where $c \in \mathbb{C} \setminus \{0\}$. We note that $\text{Vol}(P)^{-1} \int_P \ell_j dq = \ell_j(\mathbf{u}_{\text{cnt}})$.

Let us assume that \mathfrak{y} is nondegenerate. Then using also the multiplicativity of $i_{\text{qm}, (b, b(\mathfrak{y}))}^*$ ([FOOO6] Theorem 9.1) we have

$$e_{\mathfrak{y}} \cup^b \mathcal{S}^b([S_j^1]) \equiv q^{\ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}})} c e_{\mathfrak{y}} \mod q^{\ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}})} \Lambda_-^\perp.$$

Therefore by Corollary 30.5 we obtain

$$\mu_e^b([S_j^1]) = \text{Vol}(P)(\ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}})).$$

In the general case we recall that $\text{Jac}(\mathfrak{P}\mathfrak{D}_b; \mathfrak{y})$ is a local ring and the kernel of the homomorphism $\text{Jac}(\mathfrak{P}\mathfrak{D}_b; \mathfrak{y}) \rightarrow \Lambda$ defined by $[\mathfrak{P}] \mapsto \mathfrak{P}(\mathfrak{y})$ is nilpotent. Therefore

$$e_{\mathfrak{y}} \cup^b \mathcal{S}^b([S_j^1]) = a e_{\mathfrak{y}} + b$$

with $a \in \Lambda$,

$$a \equiv q^{\ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}})} c \mod q^{\ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}})} \Lambda_-^\perp$$

and b is nilpotent. We use it to show

$$\lim_{k \rightarrow \infty} \frac{\mathfrak{v}_q(e_{\mathfrak{y}} \cup^b \mathcal{S}^b([S_j^1])^k)}{k} = \ell_j(\mathbf{u}) - \ell_j(\mathbf{u}_{\text{cnt}}).$$

The proof of Theorem 30.13 is now complete. □

Remark 30.14. Theorem 30.13 also follows from Theorems 21.1 and 25.1.

REFERENCES

- [Aa] J. F. Aarnes, *Quasi-states and quasi-measure*, Adv. Math. **86** (1991), 41-67.
- [ASc1] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), 254-316.
- [ASc2] A. Abbondandolo and M. Schwarz, *Floer homology of cotangent bundles and the loop product*, Geom. Topol. **14** (2010), no. 3, 1569-1722.
- [AFOOO] M. Abouzaid, K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Quantum cohomology and split generation in Lagrangian Floer theory*, in preparation.
- [ASe] M. Abouzaid and P. Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol., **14** (2010) 627-718.
- [AM] M.I. Abreu and L. Macarini, *Remarks on Lagrangian intersections in toric manifolds*, arXiv:1105.0640.
- [Al] P. Albers, *On the extrinsic topology of Lagrangian submanifolds*, Int. Math. Res. Not. **38** (2005), 2341-2371.
- [Au] D. Auroux, *Mirror symmetry, T-duality in the complement of an anti-canonical divisor*, J. Gokova Geom Topol. **1** (2007), 51-59.
- [Ba1] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helvetici **53** (1978) 174-227.
- [Ba2] A. Banyaga, *The Structure of Classical Diffeomorphism Groups*, (1997) Kluwer Academic Publishers, Dordrecht.
- [BEP] P. Biran, V.M. Entov, L. Polterovich, *Calabi quasimorphisms for the symplectic ball*, Commun. Contemp. Math. **6** (2004), 793-802.
- [BC] P. Biran, O. Cornea, *Rigidity and uniruling for Lagrangian submanifolds*, Geom. Topol. **13** (2009), no. 5, 2881-2989.
- [Bor] M. S. Borman, *Quasi-states, quasi-morphisms, and the moment map*, arXiv:1105.1805.
- [Ca] E. Calabi, *On the group of automorphisms of a symplectic manifold*, Problems in Analysis (Lectures at the Sympos. in Honor of Salamon Bochner, Princeton Univ., Princeton NJ. 1969) Princeton University Press, New Jersey, 1970, pp 1-26.
- [CL] K. Chan and S.-C. Lau, *Open Gromov-Witten invariants and superpotentials for semi-fano toric surface*, arXiv:1010.5287.
- [Cho] C.-H. Cho, *Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle*, J. Geom. Phys. **58** (2008), 1465-1476, arXiv:0710.5454.
- [CO] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), 773-814.
- [CZ] C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. **37** (1984), 207-253.
- [ELP] Y. Eliashberg and L. Polterovich, *Symplectic quasi-state on the quadratic surface and Lagrangian submanifolds*, preprint, arXiv:1006.2501.
- [E] M. Entov, *K-area, Hofer metric and geometry of conjugacy classes in Lie groups*, Invent. Math. **146** (2001), 93-141.
- [EP1] M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. **2003**, no. 30, (2003) 1635-1676.
- [EP2] M. Entov and L. Polterovich, *Quasi-states and symplectic intersections*, Comment. Math. Helv. **81** (2006), 75-99.
- [EP3] M. Entov and L. Polterovich, *Rigid subsets of symplectic manifolds*, Compositio Math. **145** (2009), 773-826.
- [F] A. Floer, *Symplectic fixed points and holomorphic spheres*, Commun. Math. Phys. **120** (1989), 575-611.
- [FH] A. Floer and H. Hofer, *Symplectic homology. I. Open sets in \mathbb{C}^n* , Math. Z. **215** (1994), 37-88.
- [Fu1] K. Fukaya, *Floer homology for families - a progress report*, Integrable Systems, Topology, and Physics (Tokyo 2000), Contemp. Math. **309** (2002), 33-68.
- [Fu2] K. Fukaya, *Application of Floer homology of Lagrangian submanifolds to symplectic topology*, Morse theoretic methods in nonlinear analysis and in symplectic topology, 231-276, NATO Sci. Ser. II Math. Phys. Chem., **217**, Springer, Dordrecht, 2006.
- [Fu3] K. Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. **50** (2010) 521- 590, arXiv:0907.4219.

- [FOOO1] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction, I, II*, AMS/IP Studies in Advanced Mathematics, vol **46-1**, **46-2**, Amer. Math. Soc./International Press, 2009. MR2553465, MR2548482.
- [FOOO2] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds I*, Duke Math. J. **151** (2010), 23-174.
- [FOOO3] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds II; Bulk deformations*, Selecta Math. New Series, **17** no. 3, (2011), 609-711.
- [FOOO4] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Anti-symplectic involution and Floer cohomology*, submitted arXiv:0912.2646.
- [FOOO5] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Toric degeneration and non-displaceable Lagrangian tori in $S^2 \times S^2$* , to appear in Int. Math. Res. Not., doi:10.1093/imrn/rnr128, arXiv:1002.1666.
- [FOOO6] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Floer theory and mirror symmetry on toric manifolds*, preprint, arXiv:1009.1648.
- [FOOO7] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds: survey*, submitted, arXiv:1011.4044.
- [FOOO8] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Displacement of polydisks and Lagrangian Floer theory*, submitted, arXiv:1104.4267.
- [FOOO9] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory for toric surface with A_n -type singularities*, in preparation.
- [FO] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933-1048.
- [Ful] W. Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies, **131**, Princeton University Press, Princeton, 1993.
- [Gr] M. Gromov, *Pseudoholomorphic curves in symplectic geometry*, Invent. Math. **82** (1985) 304-374.
- [Gu] V. Guillemin, *Kähler structures on toric varieties*, J. Differ. Geom. **43** (1994), 285-309.
- [GLS] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge Univ. Press, Cambridge, 1966.
- [H] H. Hofer, *On the topological properties of symplectic maps*, Proc. Royal Soc. Edinburgh **115** (1990), 25-38.
- [HS] H. Hofer and D. Salamon, *Floer homology and Novikov rings*, The Floer memorial volume, 483-524, Progr. Math. **133**, Birkhäuser, Basel, 1995.
- [HZ] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Basel, 1994.
- [Iri1] H. Iritani, *Convergence of quantum cohomology by quantum Lefschetz*, J. Reine Angew. Math. **610** (2007), 29-69.
- [Iri2] H. Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. in Math. **222** (2009), 1069-1079.
- [Ku] A.G. Kushnirenko, *Polyèdres de Newton et nombre de Milnor*, Invent. Math. **32** (1976) 1 - 31.
- [LMP] F. Lalonde, D. McDuff and Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*, Invent. Math. **135** (1999) 369-385.
- [Man] Y. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications, **47**, American Mathematical Society, Providence, RI, 1999. xiv+303 pp.
- [MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 1995, Clarendon Press, Oxford.
- [MT] D. McDuff and S. Tolman, *Topological properties of Hamiltonian circle actions*, Int. Math. Res. Pap. **2006**, 72826, 1-77.
- [NNU1] T. Nishinou, Y. Nohara, and K. Ueda, *Toric degenerations of Gelfand-Cetlin systems and potential functions*, Adv. Math. **224** (2010), no. 2, 648-706.
- [NNU2] T. Nishinou, Y. Nohara, and K. Ueda, *Potential functions via toric degenerations*, preprint, arXiv:0812.0066.
- [Oh1] Y.-G. Oh, *Symplectic topology as the geometry of action functional II*, Commun. Anal. Geom. **7** (1999), 1 - 55.
- [Oh2] Y.-G. Oh, *Normalization of the Hamiltonian and the action spectrum*, J. Korean Math. Soc. **42** (2005), 65-83.

- [Oh3] Y.-G. Oh, *Chain level Floer theory and Hofer's geometry of Hamiltonian diffeomorphism group*, Asian J. Math. **9** (2002), 579-624.
- [Oh4] Y.-G. Oh, *Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds*, in "The Breadth of Symplectic and Poisson Geometry", Prog. Math. **232**, 525 – 570, Birkhäuser, Boston, 2005.
- [Oh5] Y.-G. Oh, *Spectral invariants, analysis of the Floer moduli space, and geometry of the Hamiltonian diffeomorphism group*, Duke Math. J. **130** (2005), 199-295.
- [Oh6] Y.-G. Oh, *Floer mini-max theory, the Cerf diagram, and the spectral invariants*, J. Korean Math. Soc. **46** (2009), 363-447.
- [Oh7] Y.-G. Oh, *The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows*, pp 149-177, Contemp. Math., **512**, Amer. Math. Soc., Providence, RI, 2010.
- [OM] Y.-G. Oh, S., Müller, *The group of Hamiltonian homeomorphisms and C^0 symplectic topology*, J. Symp. Geom. **5** (2007), 167 - 219.
- [OZ] Y.-G. Oh and K. Zhu, *Floer trajectories with immersed nodes and scale-dependent gluing*, J. Symp. Geom. (to appear), arXiv:0711.4187.
- [Os1] Y. Ostrover, *A comparison of Hofer's metrics on Hamiltonian diffeomorphisms and Lagrangian submanifolds*, Commun. Contemp. Math. **5** (2003), 803-811.
- [Os2] Y. Ostrover, *Calabi quasimorphisms for some non-monotone symplectic manifolds*, Algebr. Geom. Topol. **6** (2006), 405-434.
- [On] K. Ono, *On the Arnold conjecture for weakly monotone symplectic manifolds* Invent. Math. **119** (1995) 519-537.
- [Piu] S. Piunikhin, *Quantum and Floer cohomology has the same ring structure*, preprint, 1994.
- [PSS] S. Piunikhin, D. Salamon and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, In C.B. Thomas, editor, Contact and Symplectic Geometry, vol **8** of Publ. Newton Institute, pp. 171-200, Cambridge Univ. Press, Cambridge, 1996.
- [PS] G. Polya and G. Szegő, *Problems and theorems in Analysis*, Vol **1**, Springer-Verlag, Berlin, 1972.
- [P] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures in Math. ETH Zürich, Birkhäuser, Basel, 2001.
- [RT] Y. Ruan and G. Tian, *Bott-type symplectic Floer cohomology and its multiplicative structure*, Math. Res. Lett. **2** (1995) 203-219.
- [SW] D. Salamon and J. Weber, *Floer homology and the heat flow*, Geom. Funct. Anal. **16** (2006), 1050-1138
- [Sc1] M. Schwarz, *Cohomology operations from S^1 -cobordisms in Floer homology*, Ph.D. thesis, Swiss Federal Inst. of Techn. Zürich, Diss ETH No. 11182, 1995.
- [Sc2] M. Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), 419-461.
- [Se] P. Seidel, π_1 of symplectic automorphism groups and invertibles in quantum homology rings, Geom. Funct. Anal. **7** (1997), no. 6, 1046-1095.
- [Si] J.-C. Sikorav, *Some properties of holomorphic curves in almost complex manifolds*, Holomorphic curves in symplectic geometry, ed. M. Audin, Progress in Math. **117**, 165-190 (1994) Birkhäuser, Basel.
- [Us1] M. Usher, *Spectral numbers in Floer theories*, Compos. Math. **144** (2008), 1581-1592.
- [Us2] M. Usher, *Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds*, Israel J. Math **184** (2011), 1-57.
- [Us3] M. Usher, *Duality in filtered Floer-Novikov complexes*, J. Topol. Anal. **2** (2010), no. 2, 233-258.
- [Us4] M. Usher, *Deformed Hamiltonian Floer theory, Capacity Estimate, and Calabi-quasimorphisms*, Geom. Topol. **15** (2011), 1313-1417.
- [Vi1] C. Viterbo, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292** (1992), 685-710.
- [Vi2] C. Viterbo, *Functors and computations in Floer homology and applications, Part I*, Geom. Funct. Anal., **9** (1999) 985-1033.

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